

Complexity Theory

Part Two

Recap from Last Time

The Complexity Class **P**

- The complexity class **P** (*polynomial time*) is defined as

$$\mathbf{P} = \{ L \mid \text{There is a polynomial-time decider for } L \}$$

- Intuitively, **P** contains all decision problems that can be solved efficiently.
- This is like class **R**, except with “efficiently” tacked onto the end.

The Complexity Class **NP**

- The complexity class **NP** (*nondeterministic polynomial time*) contains all problems that can be verified in polynomial time.
- Formally:

$$\mathbf{NP} = \{ L \mid \text{There is a polynomial-time verifier for } L \}$$

- Intuitively, **NP** is the set of problems where “yes” answers can be checked efficiently.
- This is like the class **RE**, but with “efficiently” tacked on to the definition.

The Biggest Unsolved Problem in
Theoretical Computer Science:

$$\mathbf{P} \stackrel{?}{=} \mathbf{NP}$$

New Stuff!

P = { L | there is a polynomial-time decider for L }

NP = { L | there is a polynomial-time verifier for L }

R = { L | there is a ~~polynomial-time~~ decider for L }

RE = { L | there is a ~~polynomial-time~~ verifier for L }

We know that $\mathbf{R} \neq \mathbf{RE}$.

So does that mean $\mathbf{P} \neq \mathbf{NP}$?

A Problem

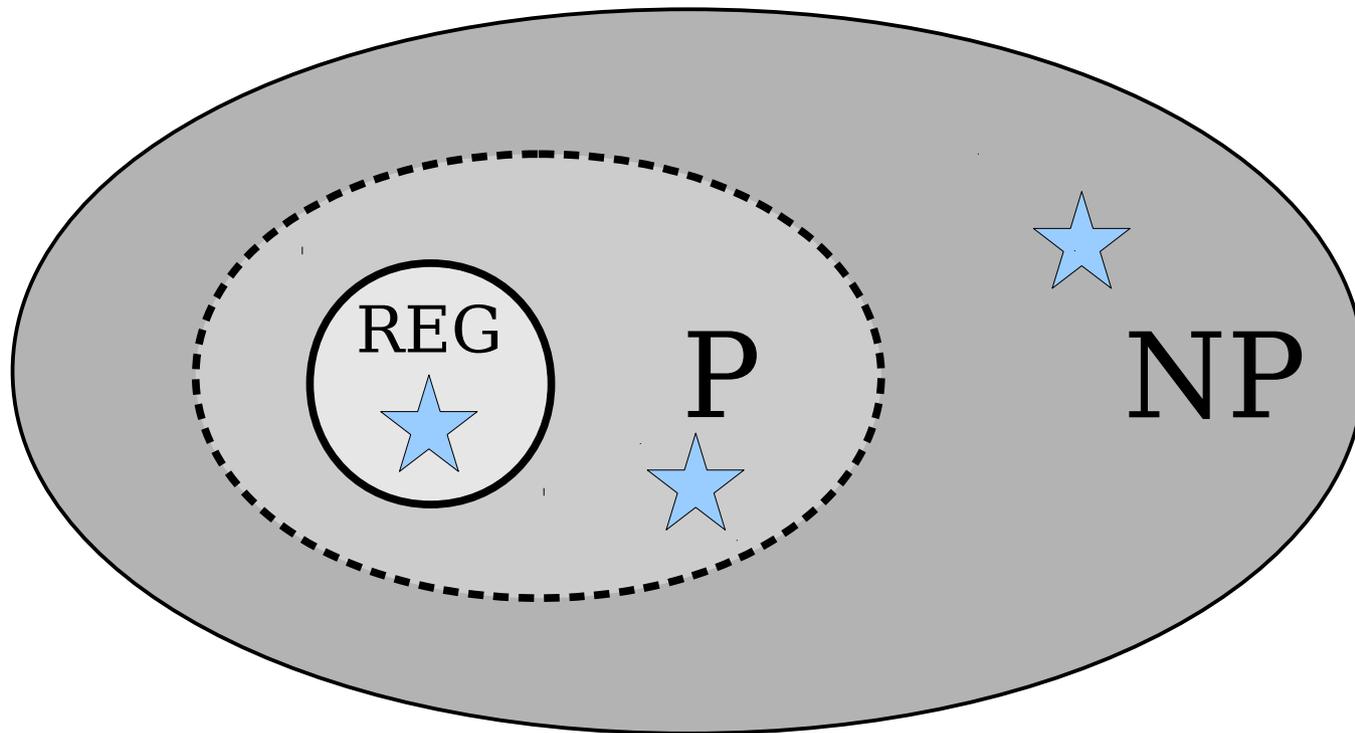
- The **R** and **RE** languages correspond to problems that can be decided and verified, *period*, without any time bounds.
- To reason about what's in **R** and what's in **RE**, we used two key techniques:
 - **Universality**: TMs can simulate other TMs.
 - **Self-Reference**: TMs can get their own source code.
- Why can't we just do that for **P** and **NP**?

Theorem (Baker-Gill-Solovay): Any proof that purely relies on universality and self-reference cannot resolve $\mathbf{P} \stackrel{?}{=} \mathbf{NP}$.

Proof: Take CS154!

So how *are* we going to
reason about **P** and **NP**?

A Challenge



Problems in **NP** vary widely in their difficulty, even if $\mathbf{P} = \mathbf{NP}$.

How can we rank the relative difficulties of problems?

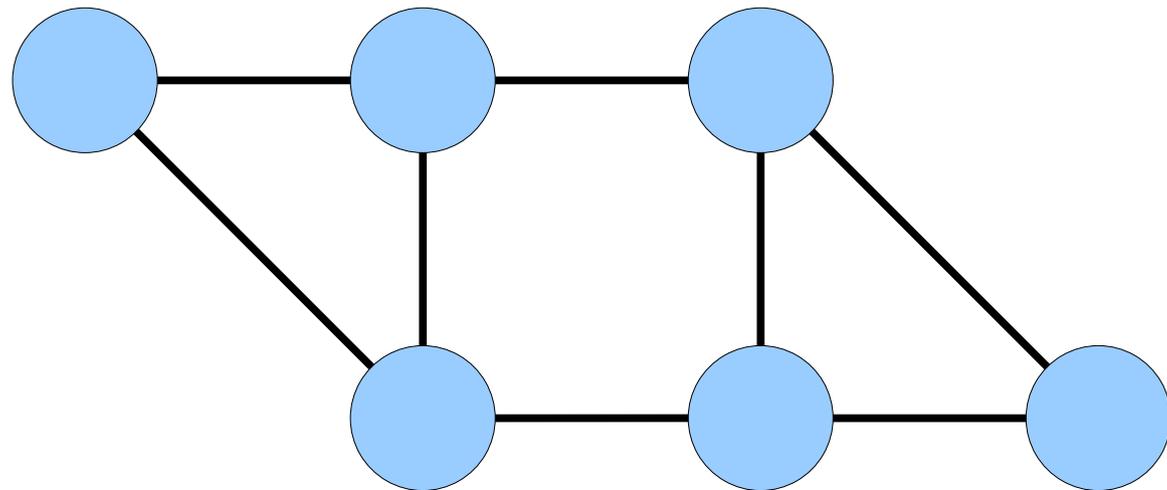
Reducibility

Maximum Matching

- Given an undirected graph G , a ***matching*** in G is a set of edges such that no two edges share an endpoint.
- A ***maximum matching*** is a matching with the largest number of edges.

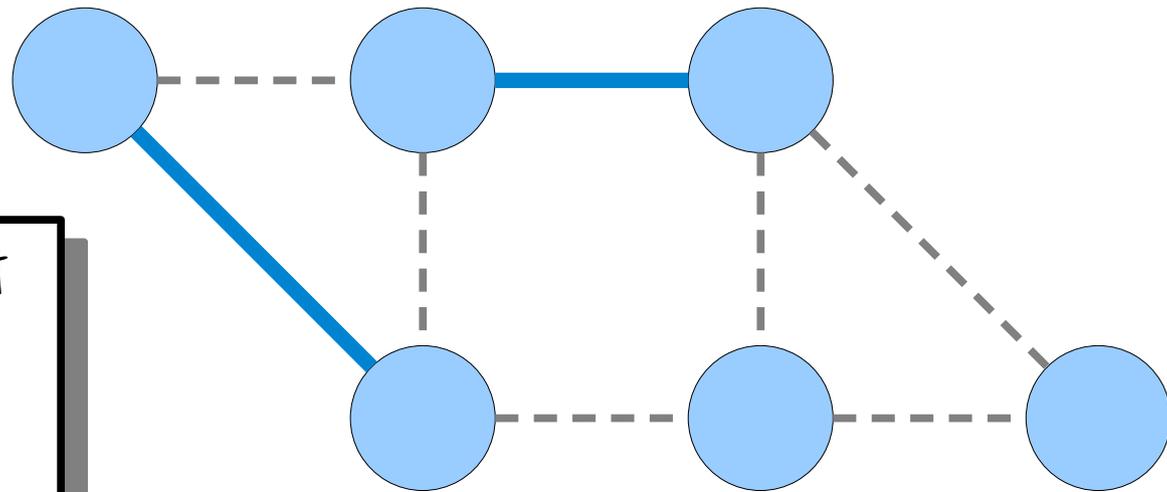
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Maximum Matching

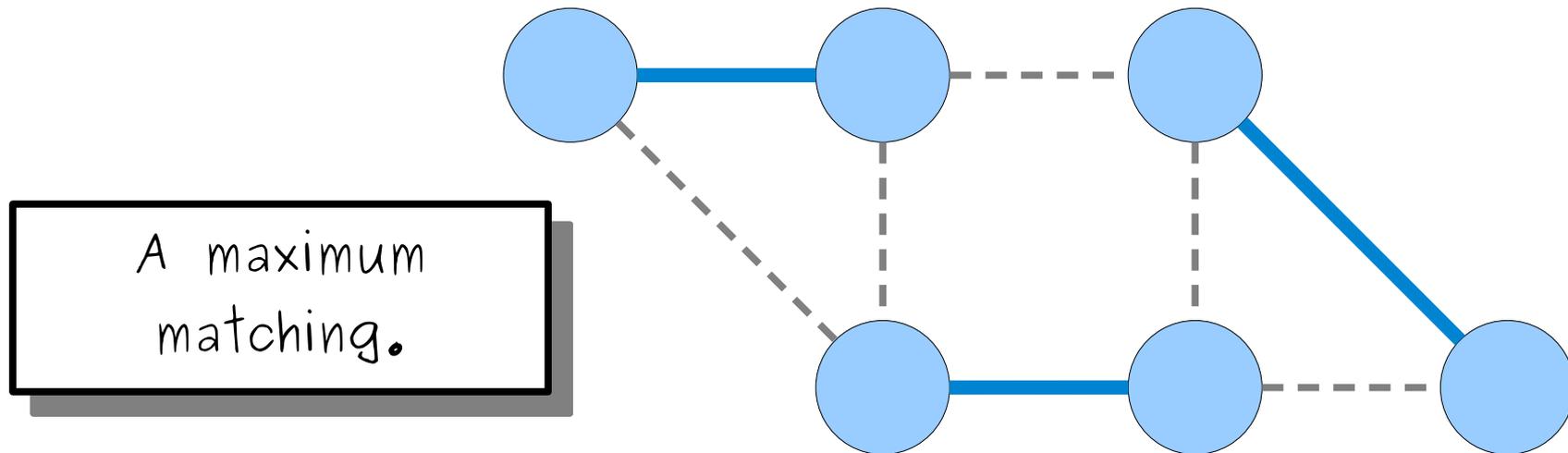
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A matching, but
not a maximum
matching.

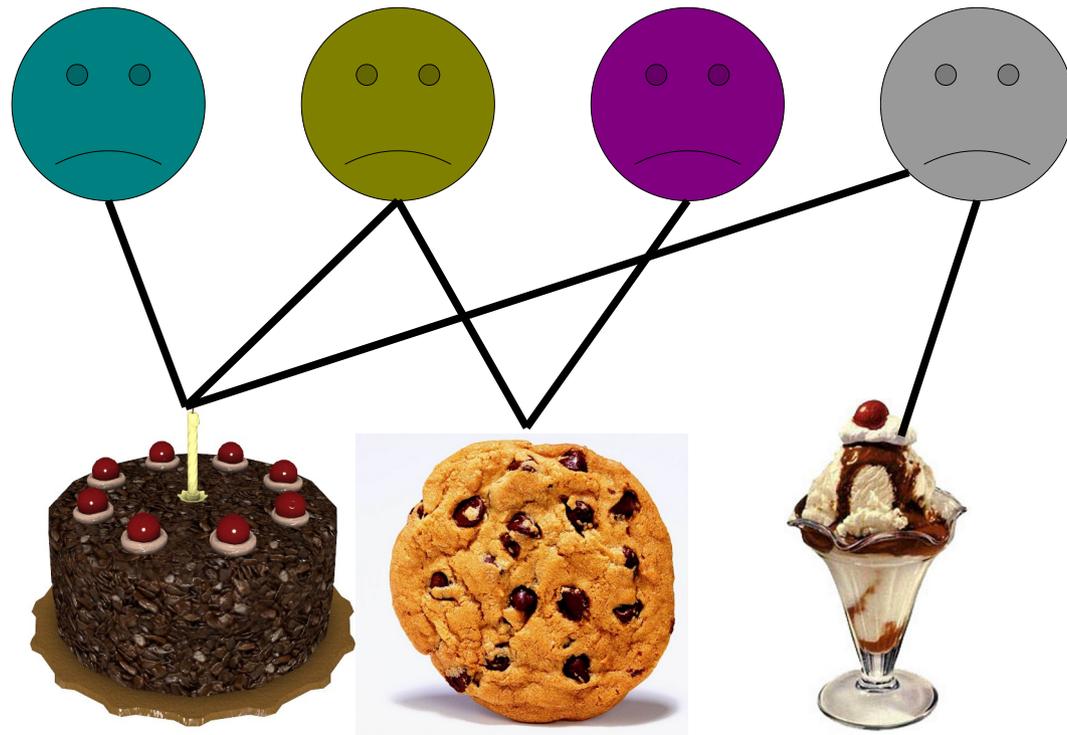
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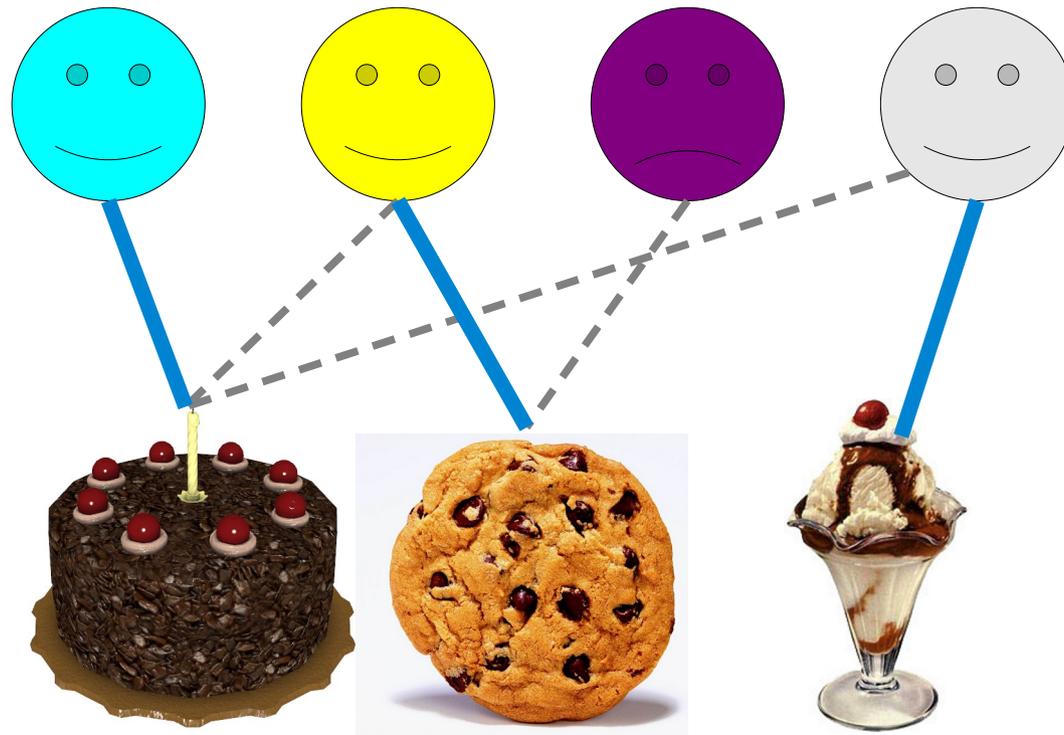
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Maximum Matching

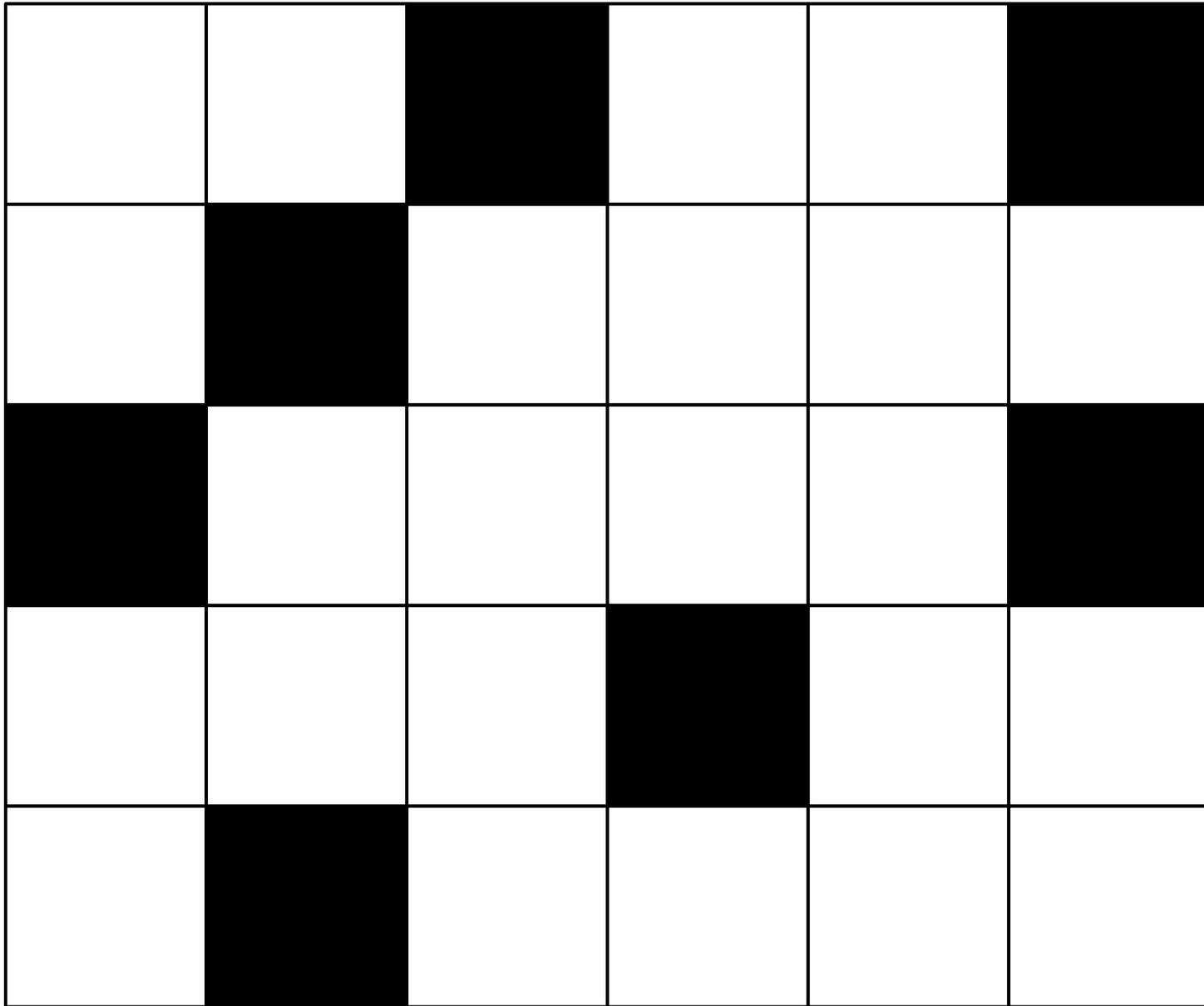
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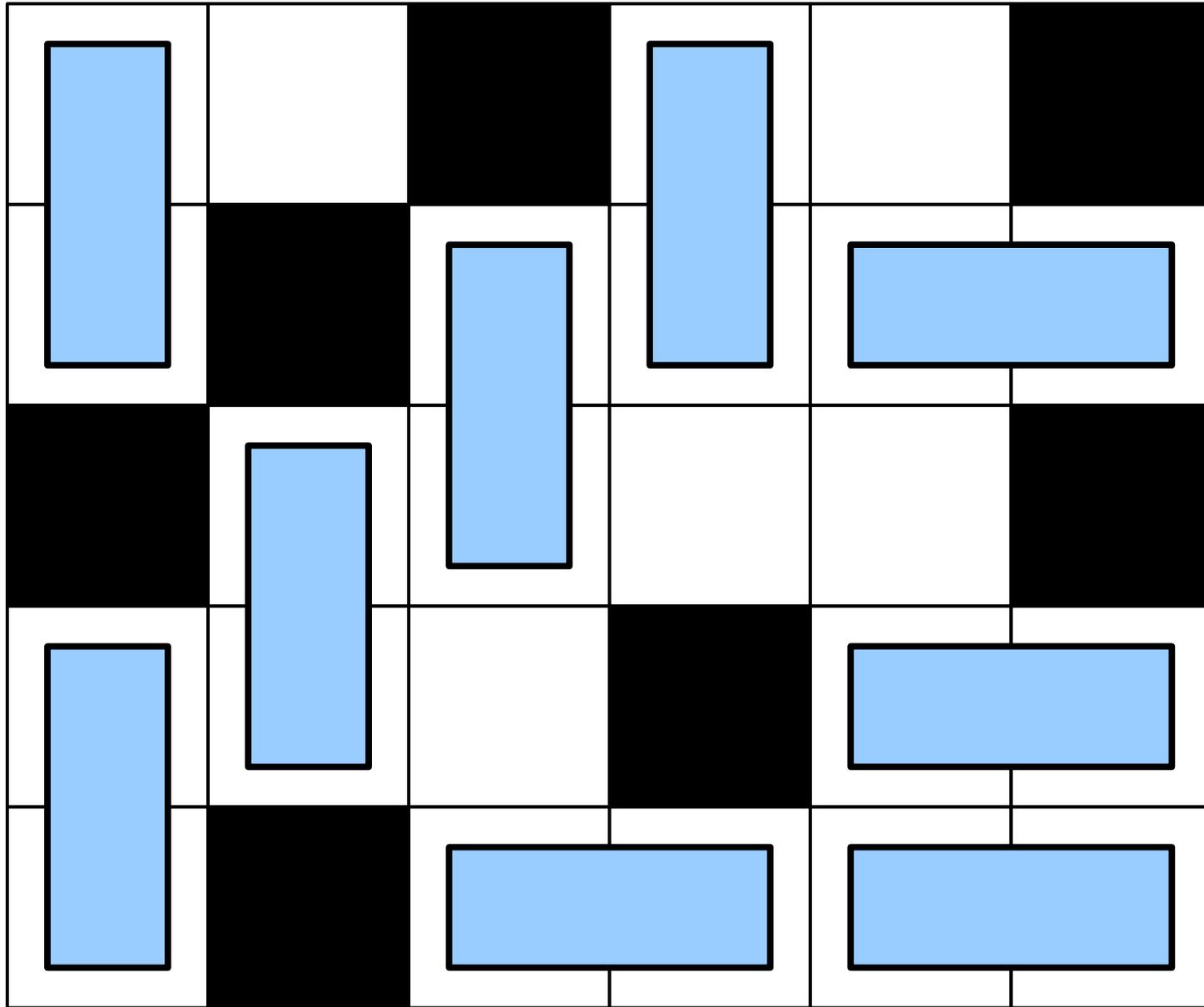
Maximum Matching

- Jack Edmonds' paper “Paths, Trees, and Flowers” gives a polynomial-time algorithm for finding maximum matchings.
 - He's the guy from last time with the quote about “better than decidable.”
- Using this fact, what other problems can we solve?

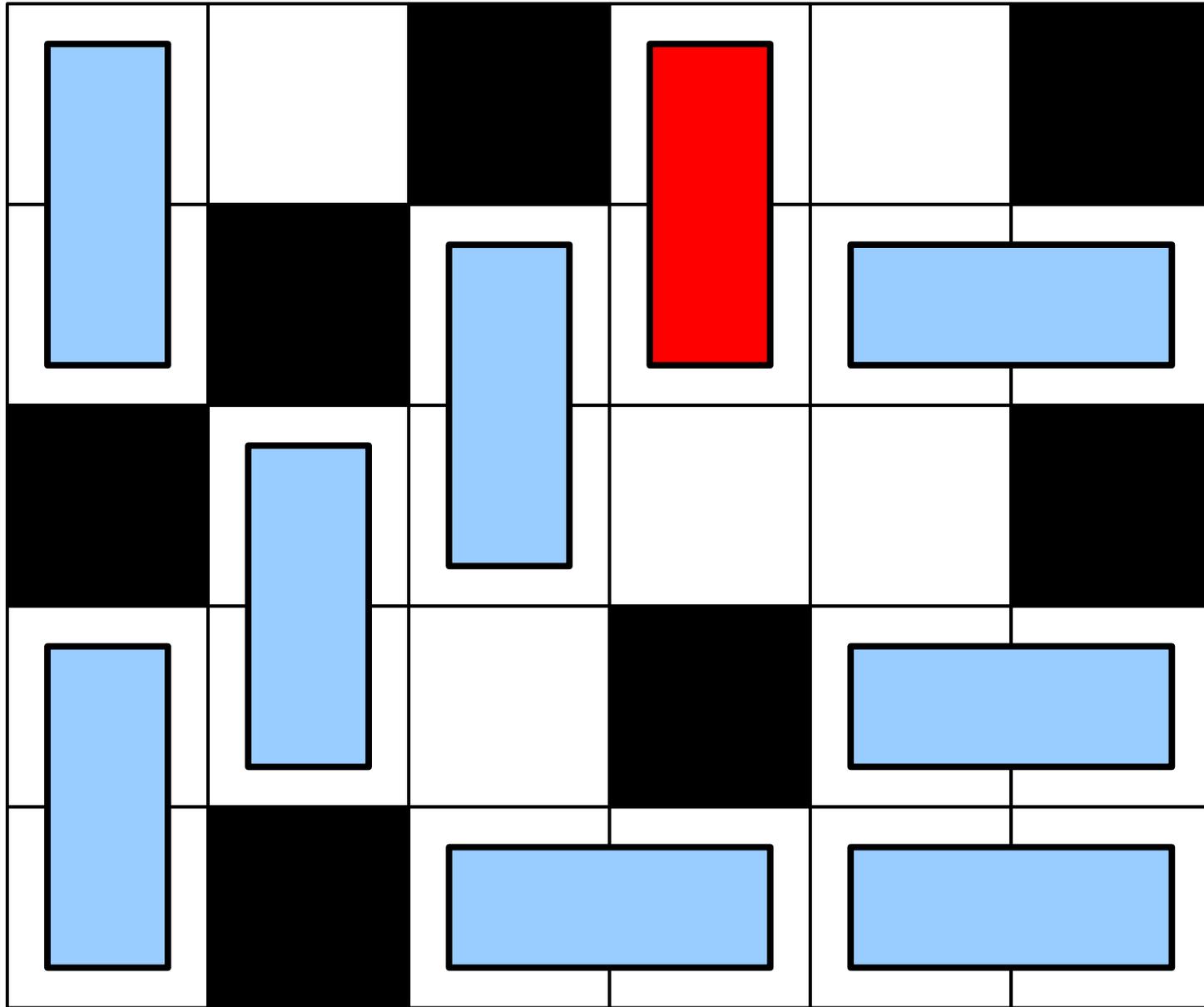
Domino Tiling



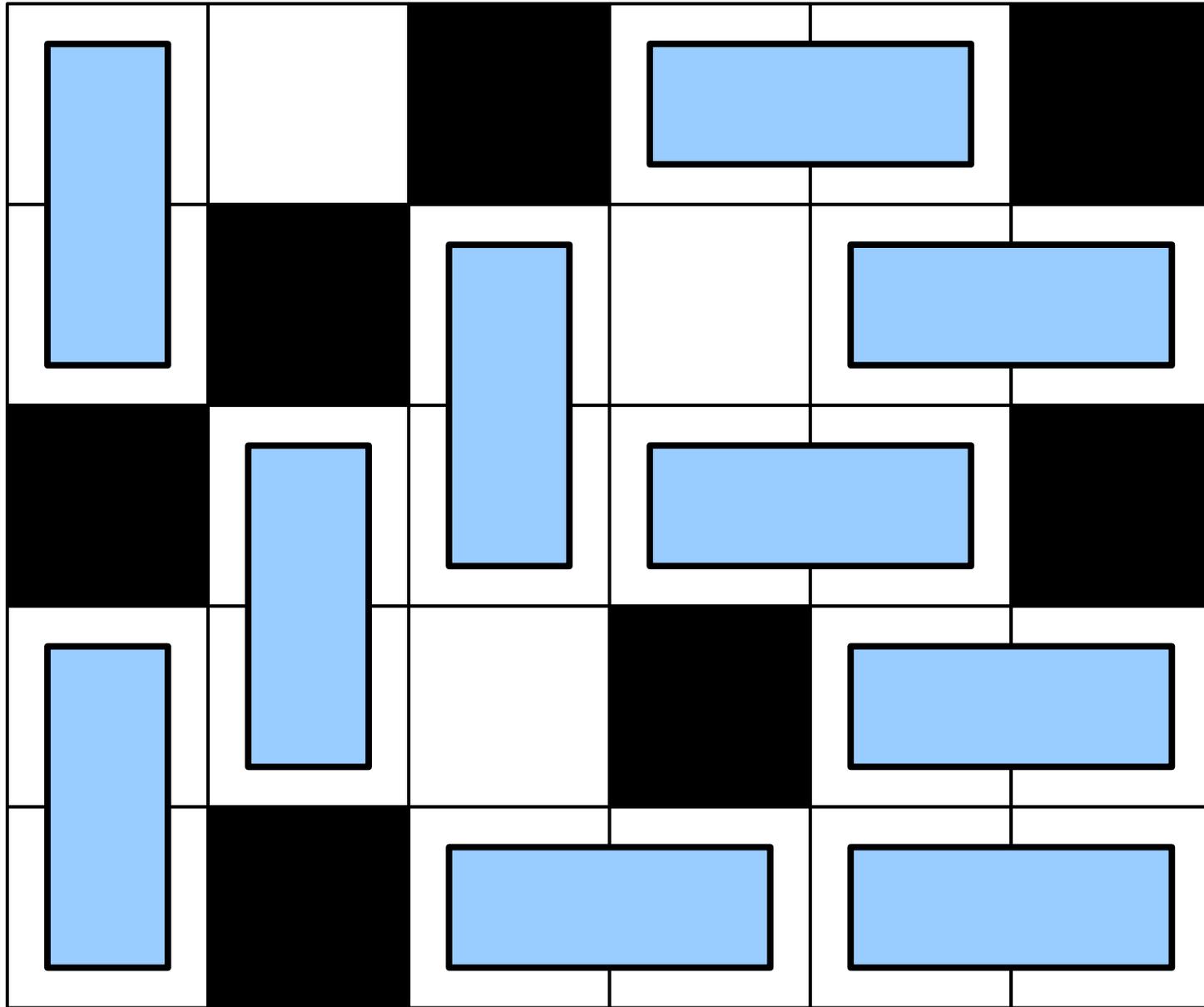
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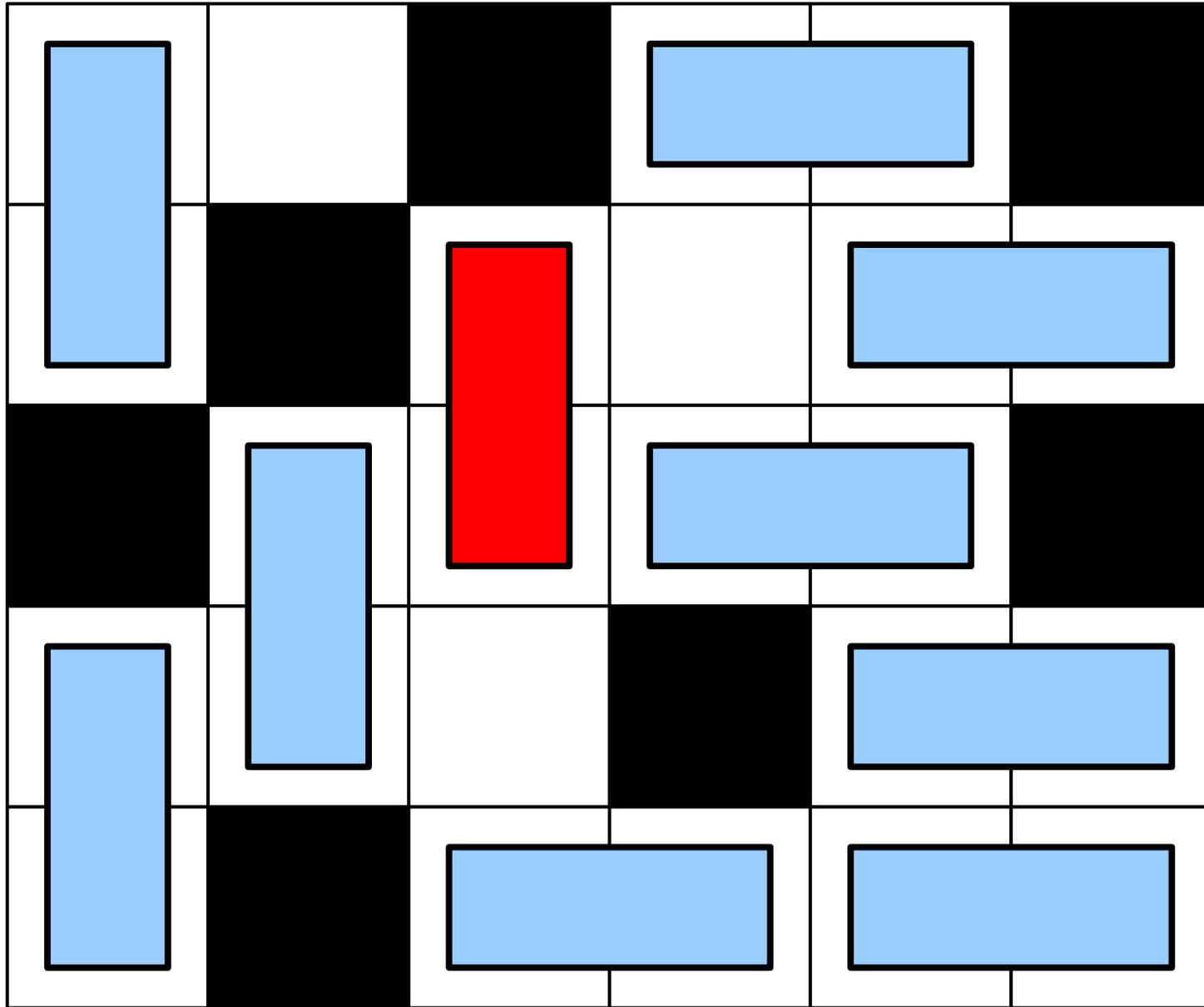
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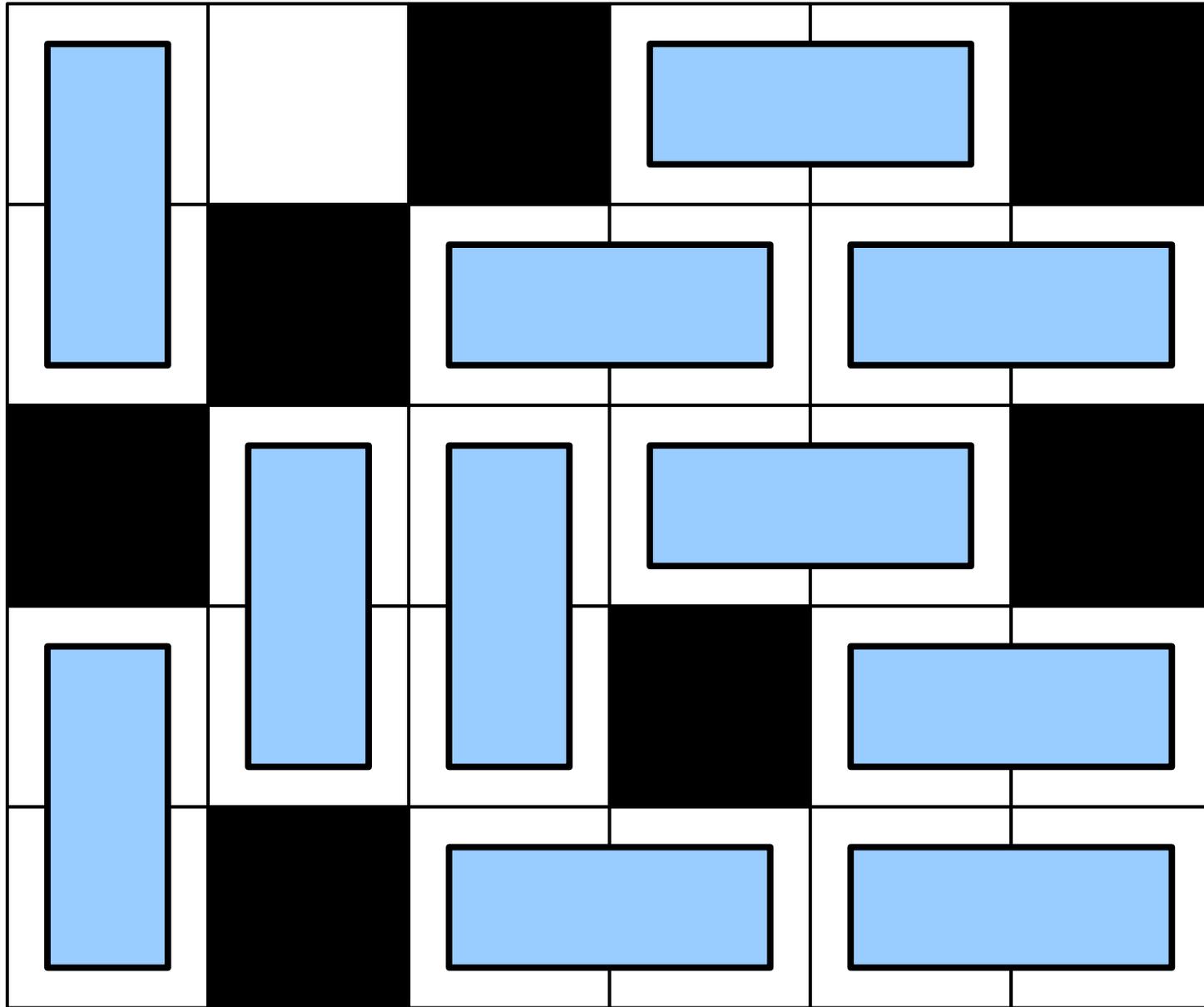
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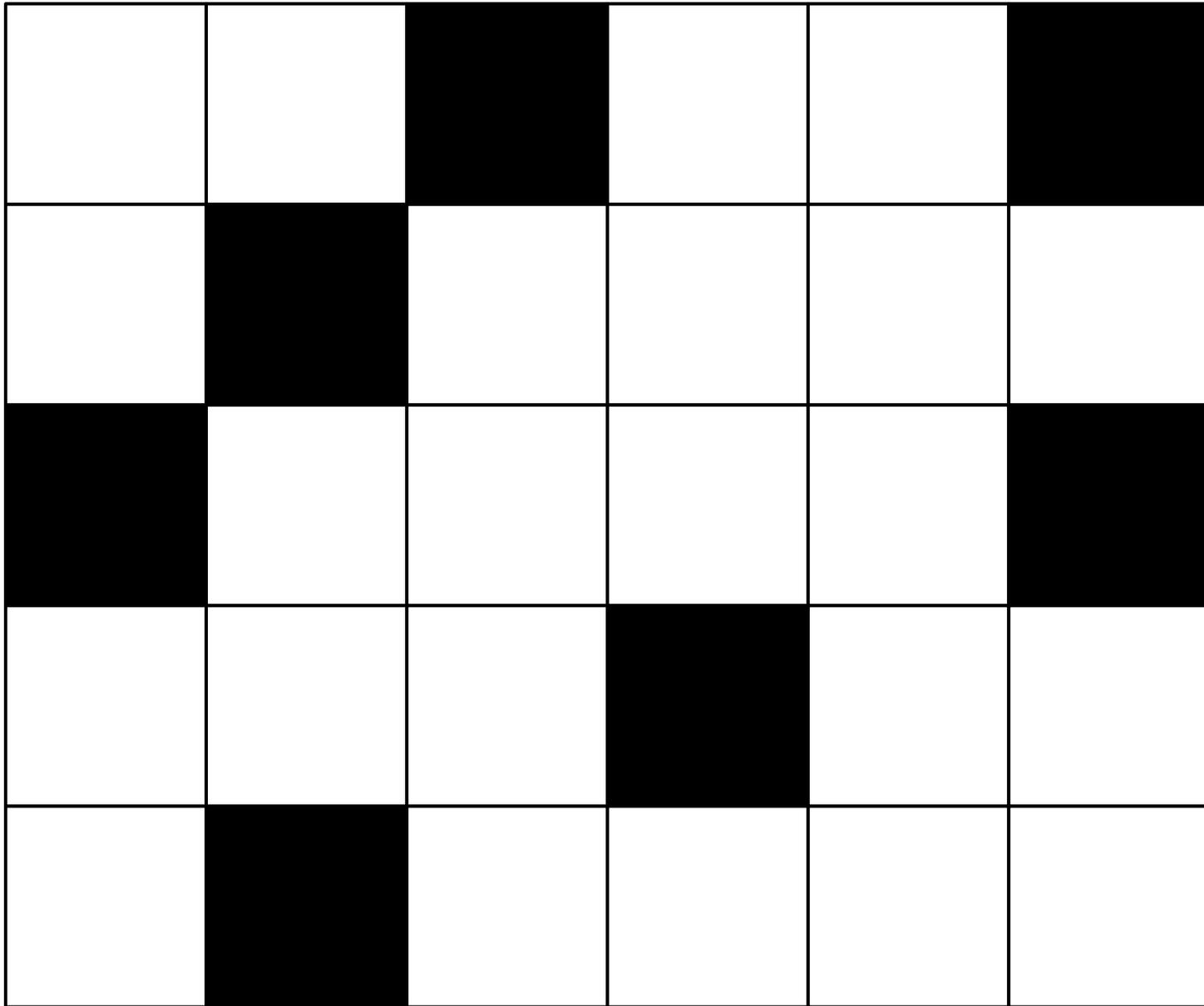
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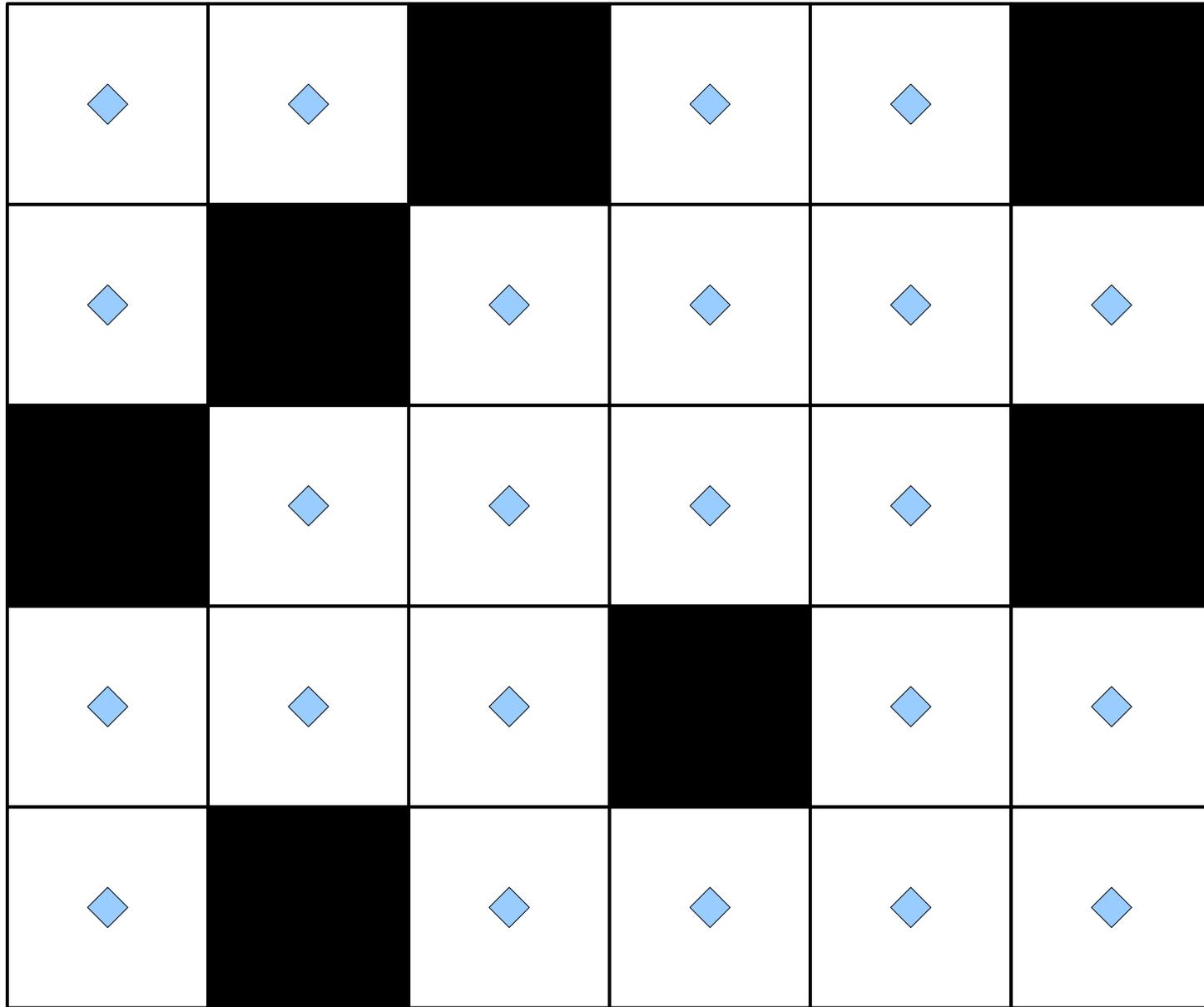
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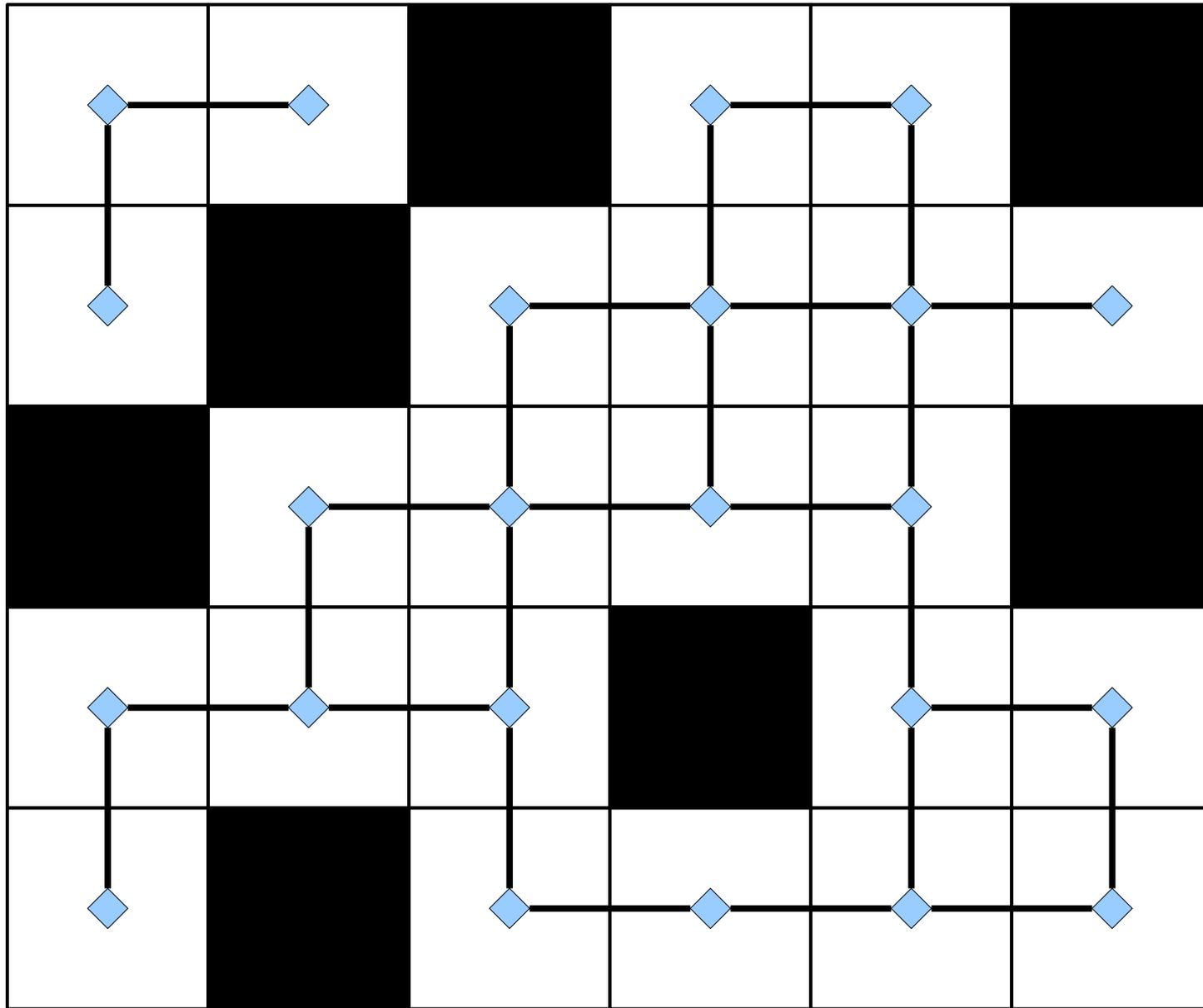
Solving Domino Tiling



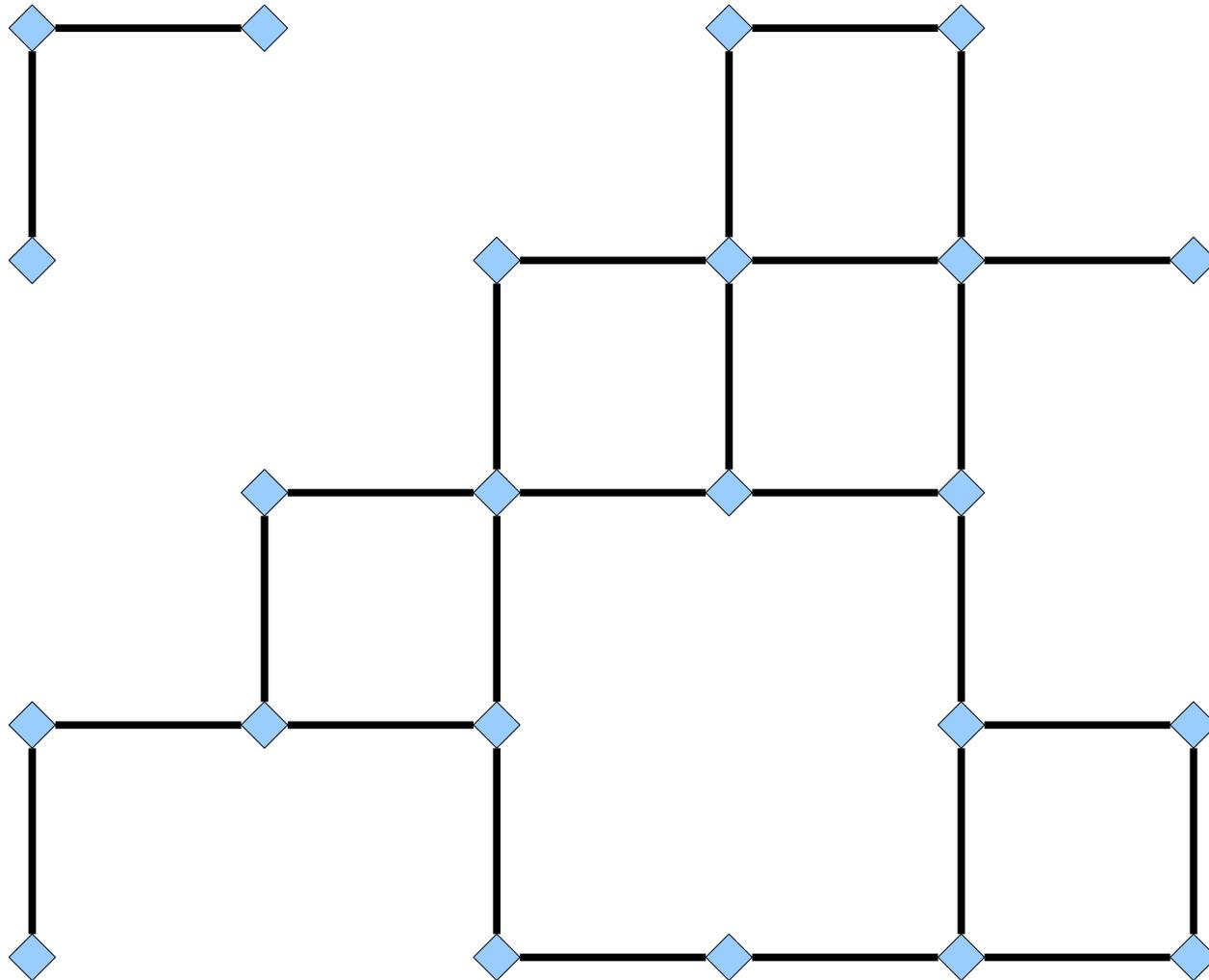
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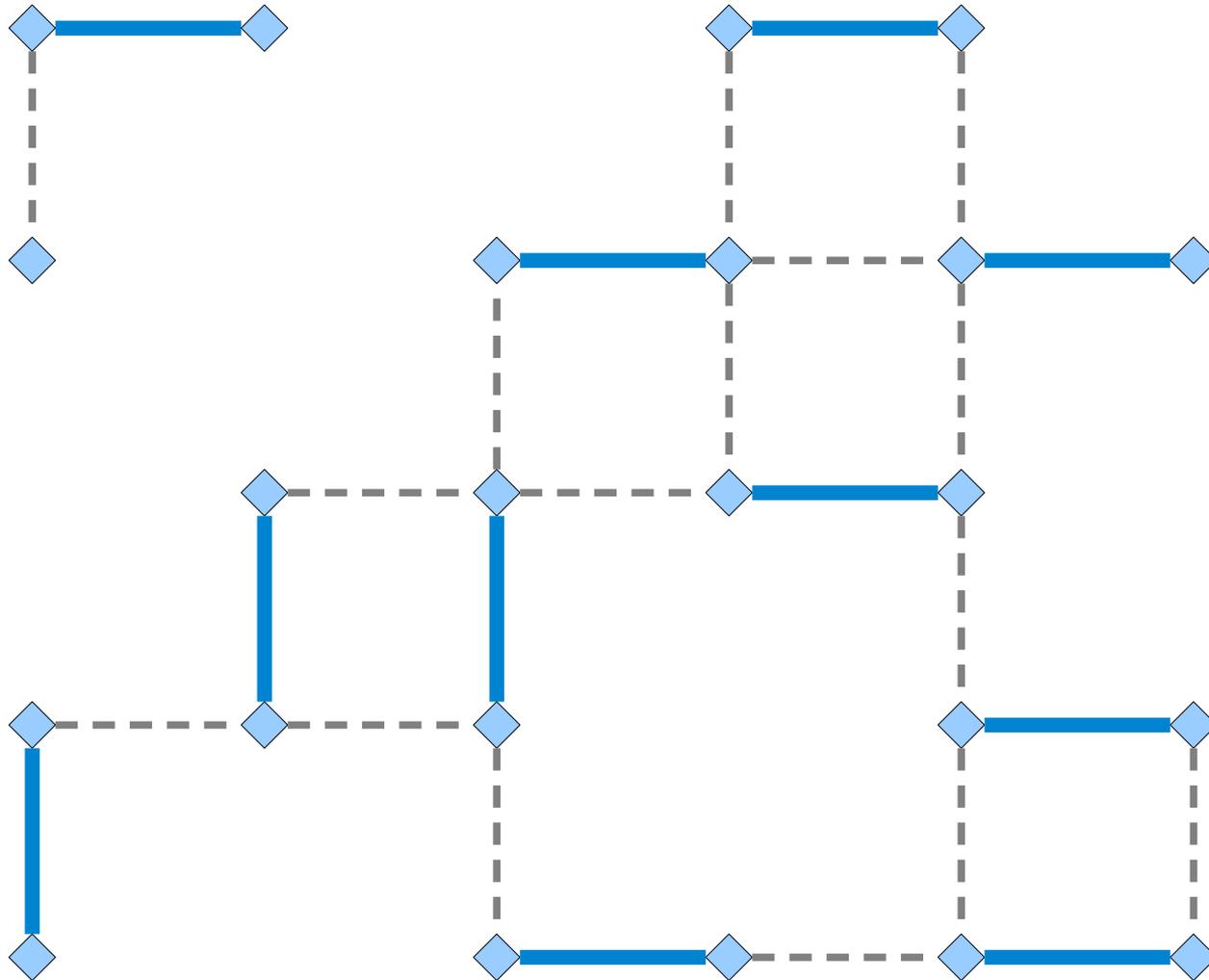
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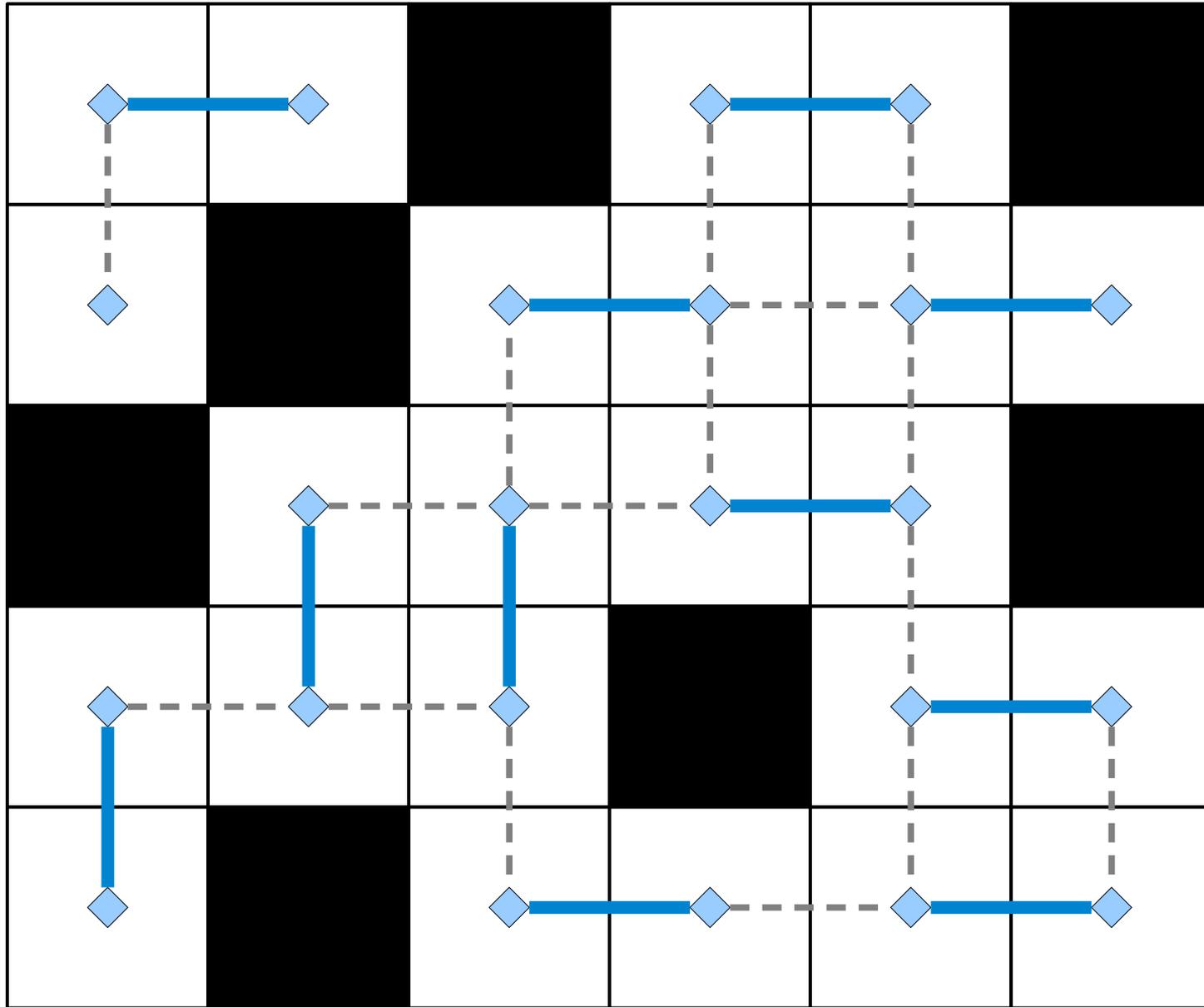
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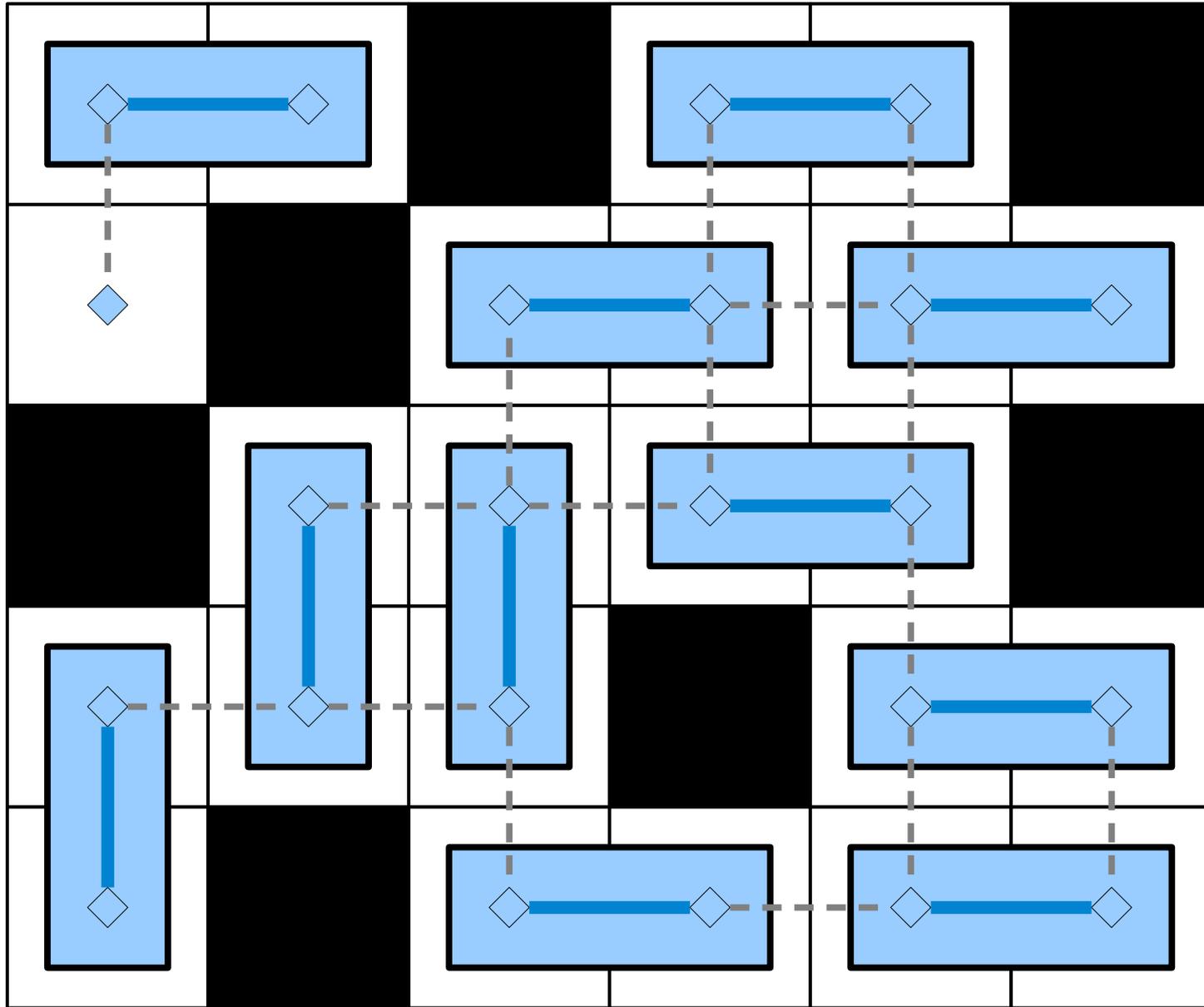
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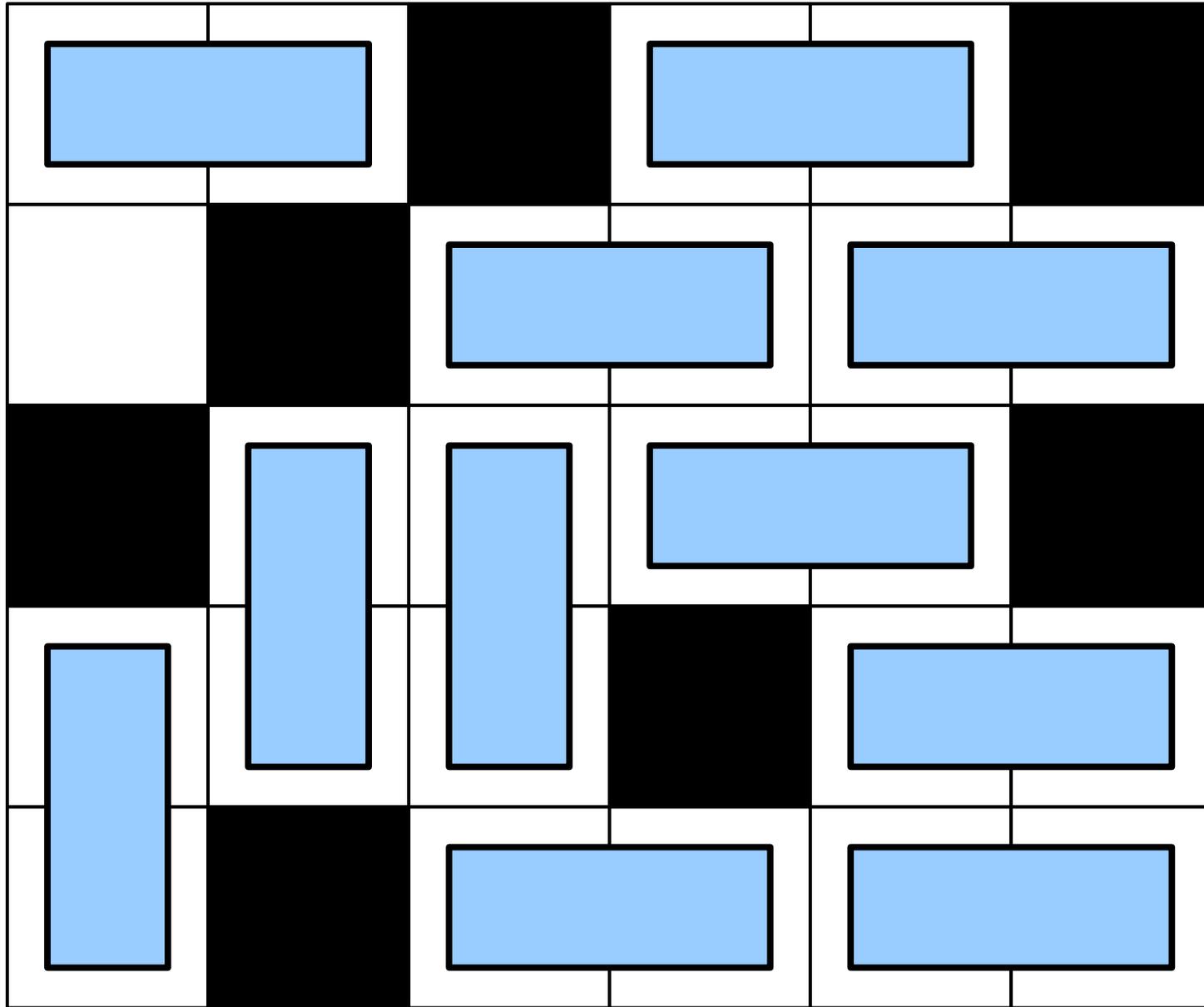
Solving Domino Tiling



Solving Domino Tiling



Solving Domino Tiling



```
bool canPlaceDominoes(Grid G, int k) {  
    return hasMatching(gridToGraph(G), k);  
}
```

Which of the following is the most reasonable conclusion to draw, given the existence of the above function?

- A. Solving domino tiling on a 2D grid can't be "harder" than solving maximum matching.
- B. Solving maximum matching can't be "harder" than solving domino tiling on a 2D grid.
- C. Both A and B.

Answer at

<https://cs103.stanford.edu/pollev>

Intuition:

Tiling a grid with dominoes can't be “harder” than solving maximum matching, because if we can solve maximum matching efficiently, we can solve domino tiling efficiently.

Another Example

Satisfiability

- A propositional logic formula φ is called **satisfiable** if there is some assignment to its variables that makes it evaluate to true.
- Which of the following formulas are satisfiable?

$$p \wedge q$$

$$p \wedge \neg p$$

$$p \rightarrow (q \wedge \neg q)$$

- An assignment of true and false to the variables of φ that makes it evaluate to true is called a **satisfying assignment**.

SAT

- The **boolean satisfiability problem** (**SAT**) is the following:

Given a propositional logic formula φ , is φ satisfiable?

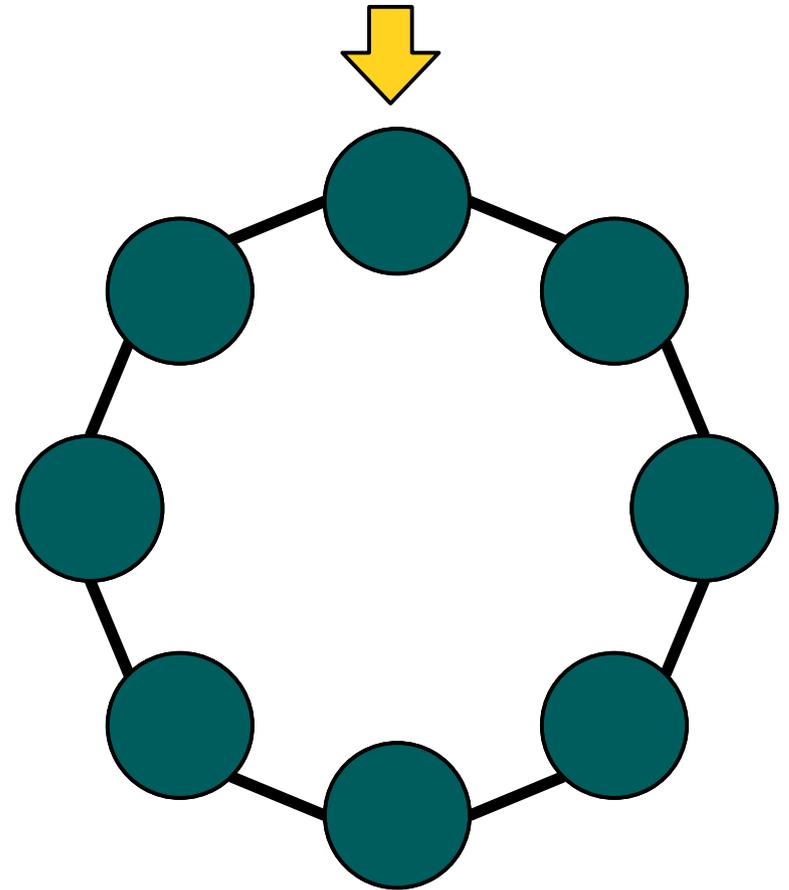
- Formally:

$SAT = \{ \langle \varphi \rangle \mid \varphi \text{ is a satisfiable PL formula} \}$

- Finding good algorithms for SAT is an active area of research for reasons we'll discuss later today.
- We have some pretty decent algorithms for solving SAT reasonably quickly most of the time.
- Given this, what other problems can we solve?

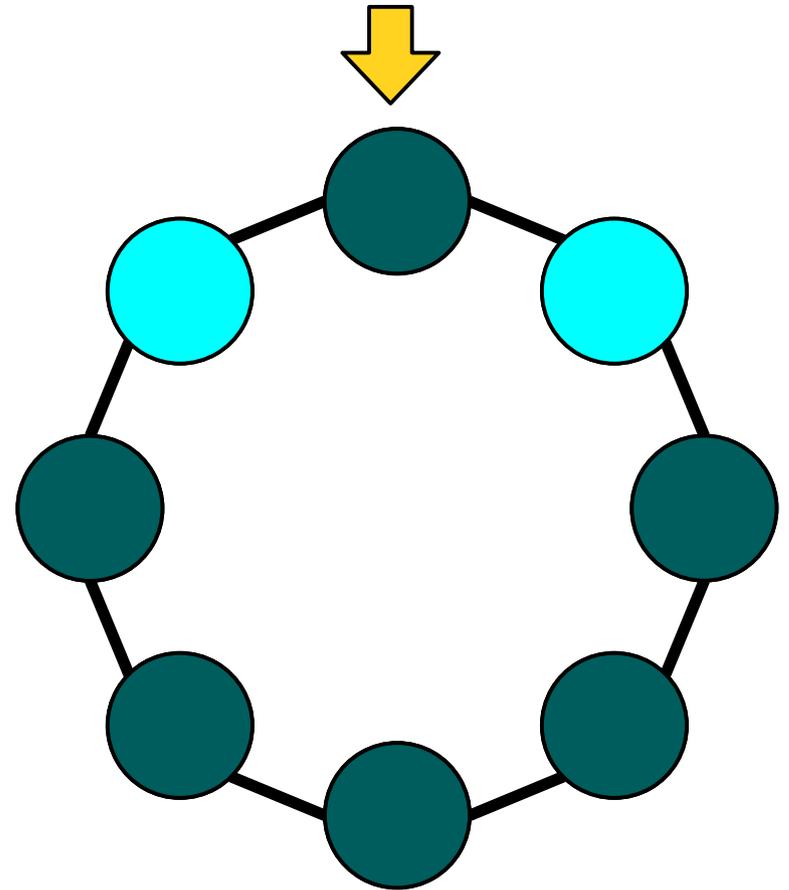
Lights Out

- You're given a ring of pushbuttons. Each pushbutton has a light that is either ON or OFF.
- If you push a button, it toggles the state of the two adjacent lights in the ring. (Lights that are ON turn OFF and vice-versa.)



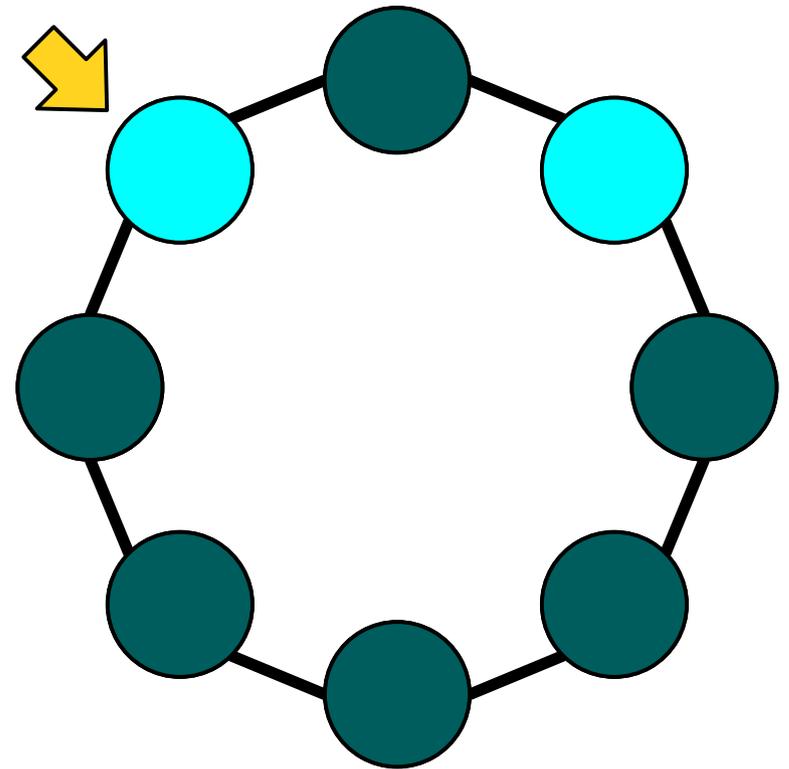
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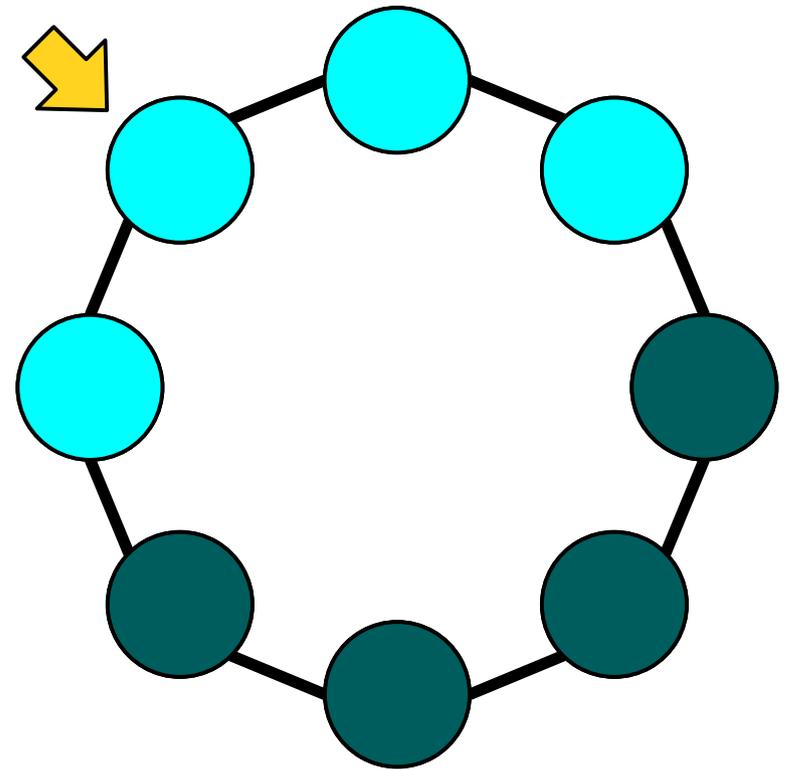
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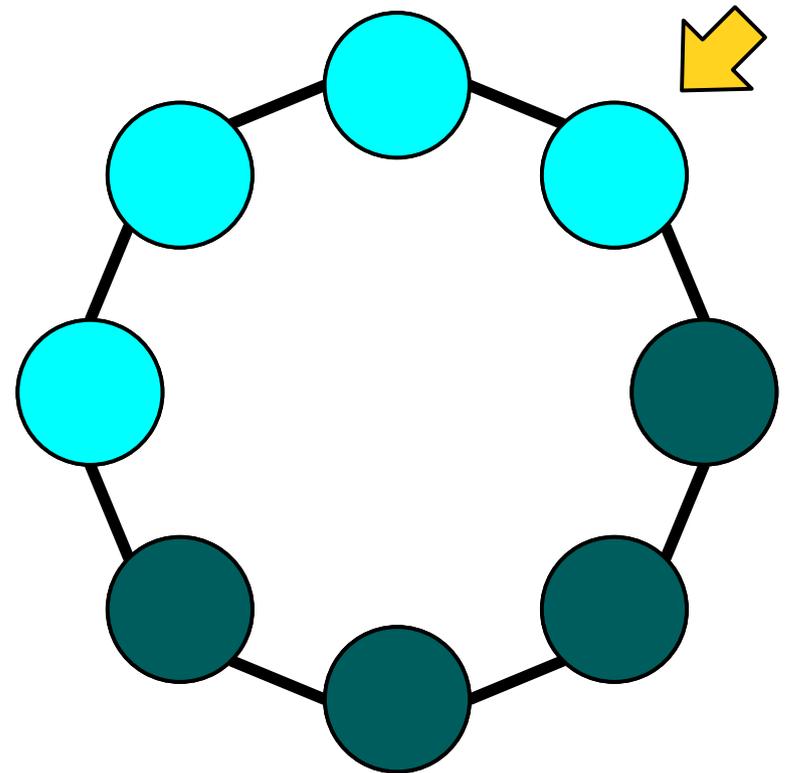
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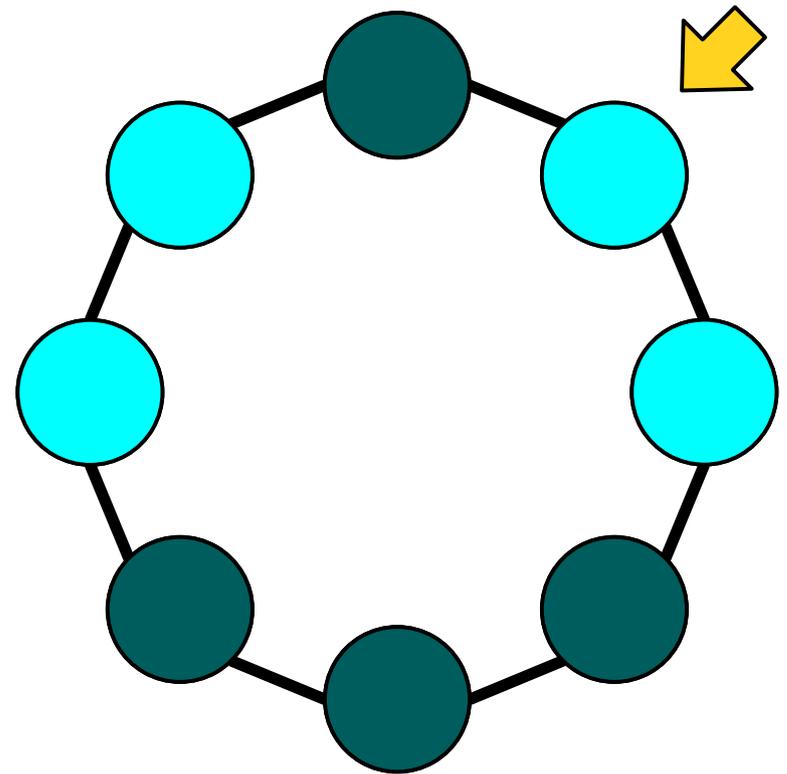
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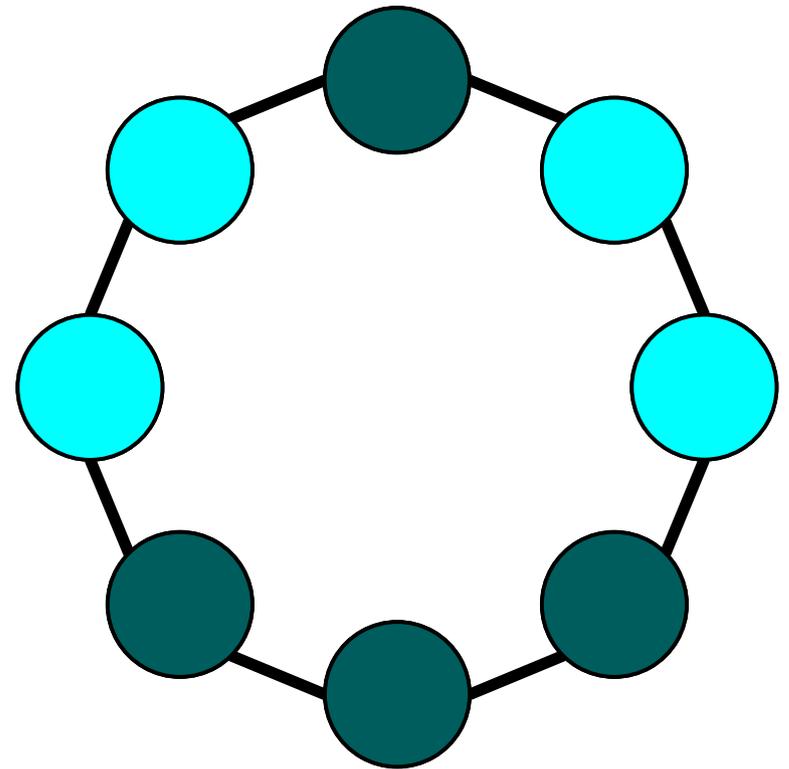
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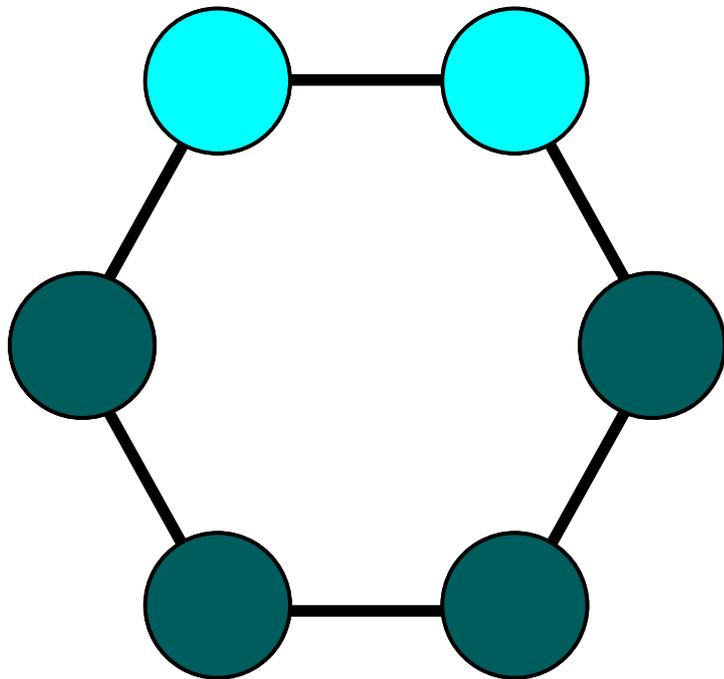
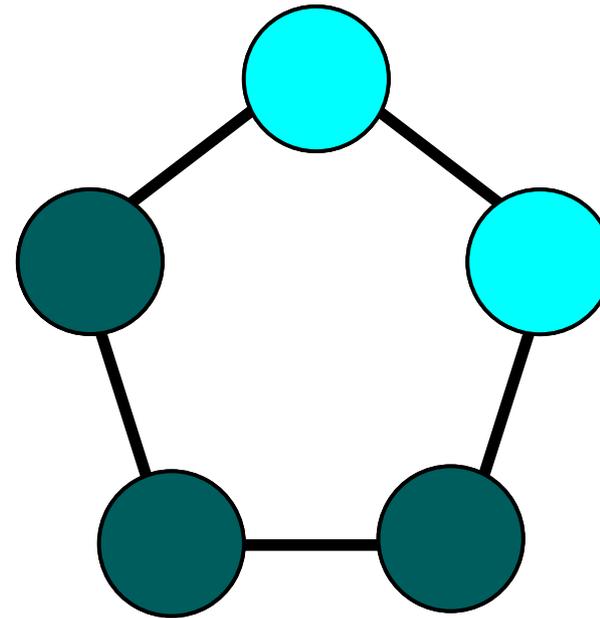
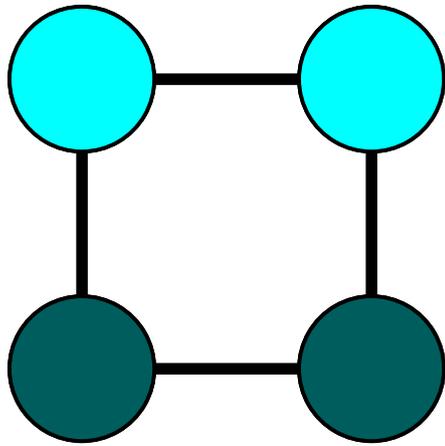
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- **Question:** Given an initial configuration of lights, can you turn all the lights off?

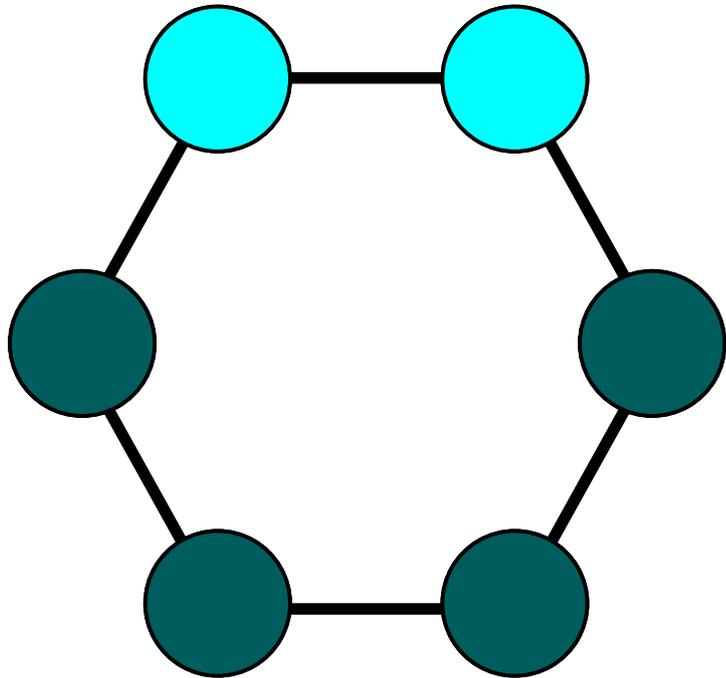
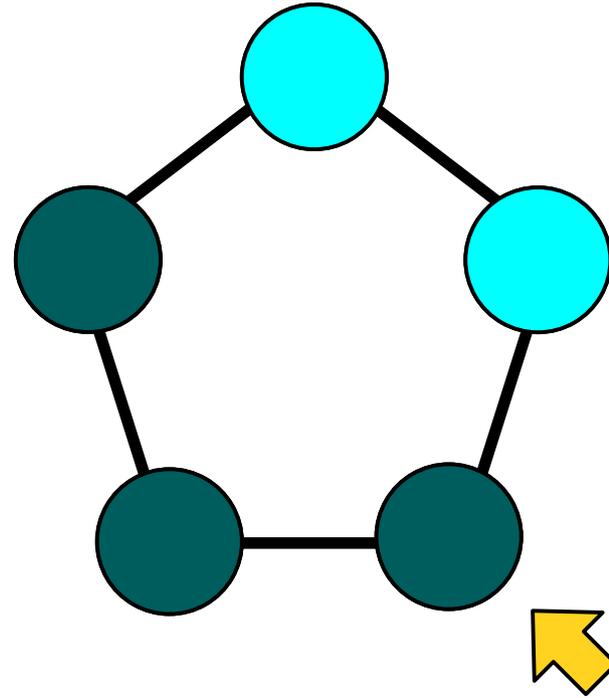
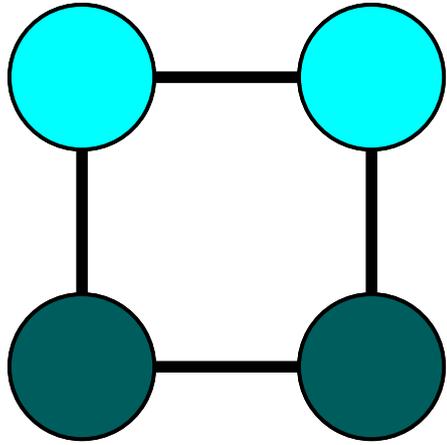


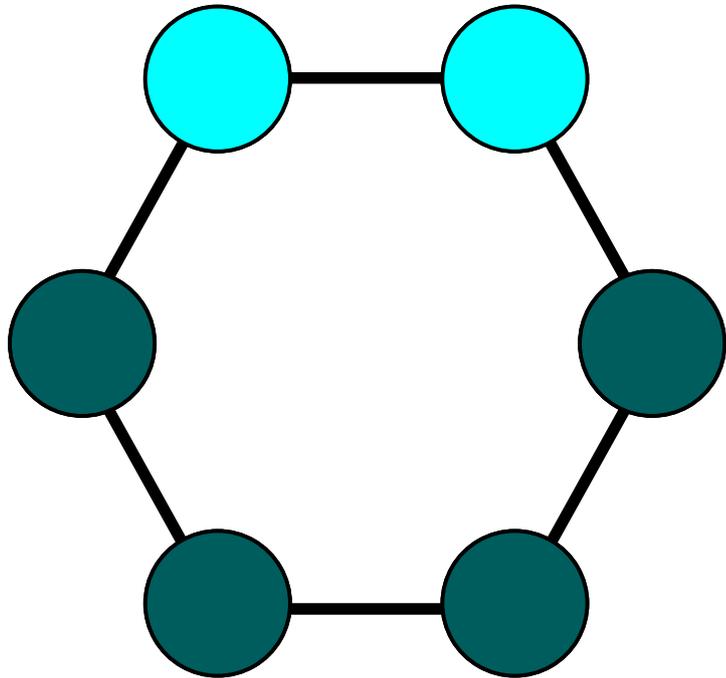
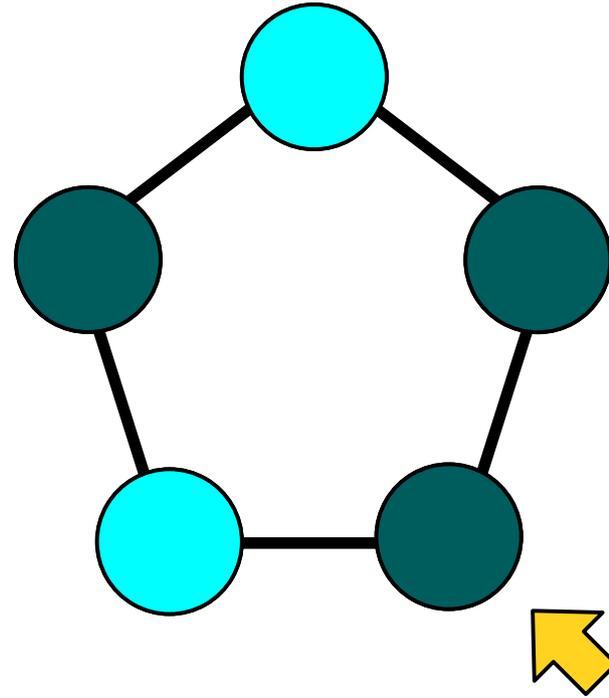
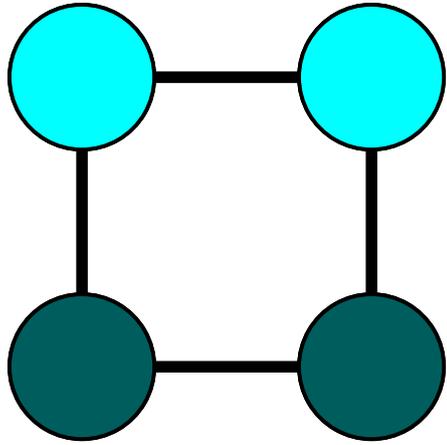


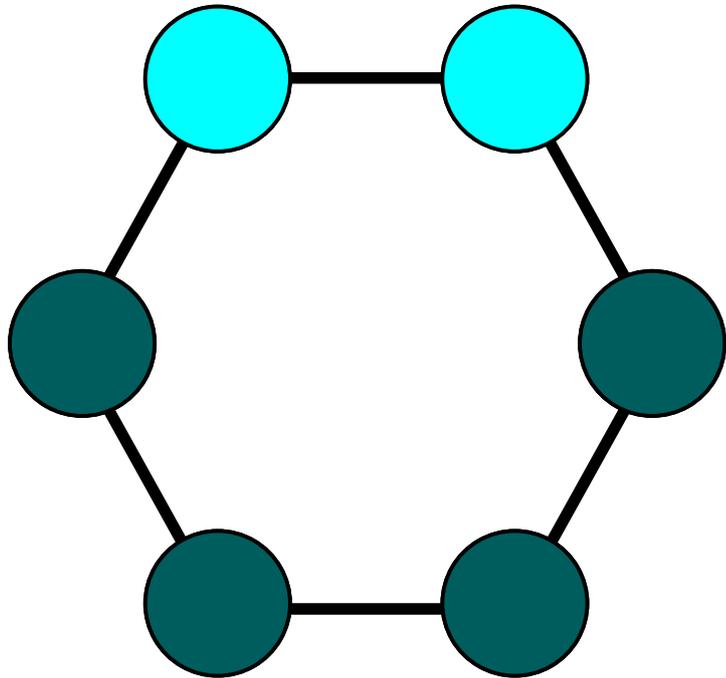
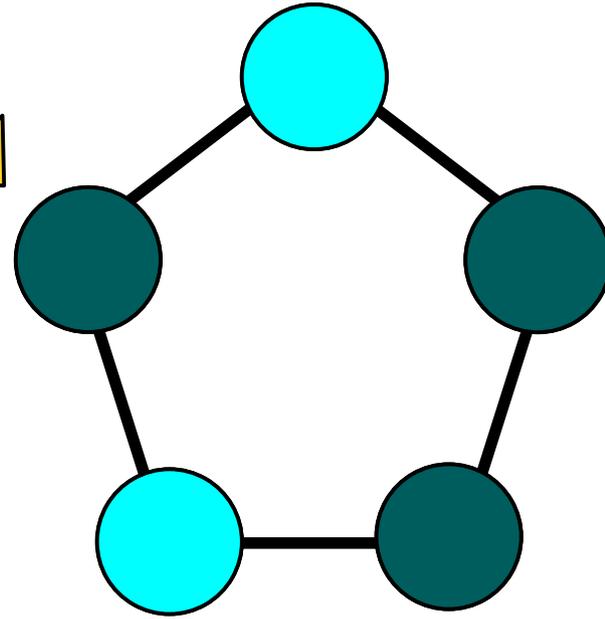
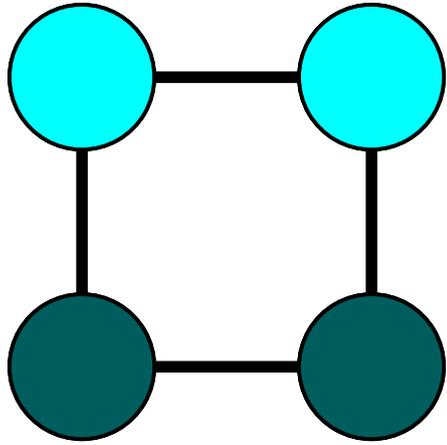
In which of these rings can you
turn off all the lights?

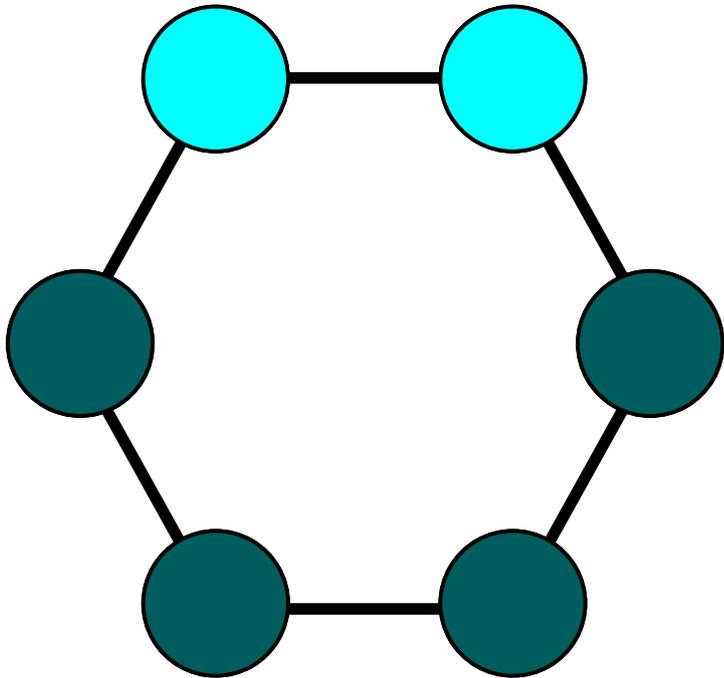
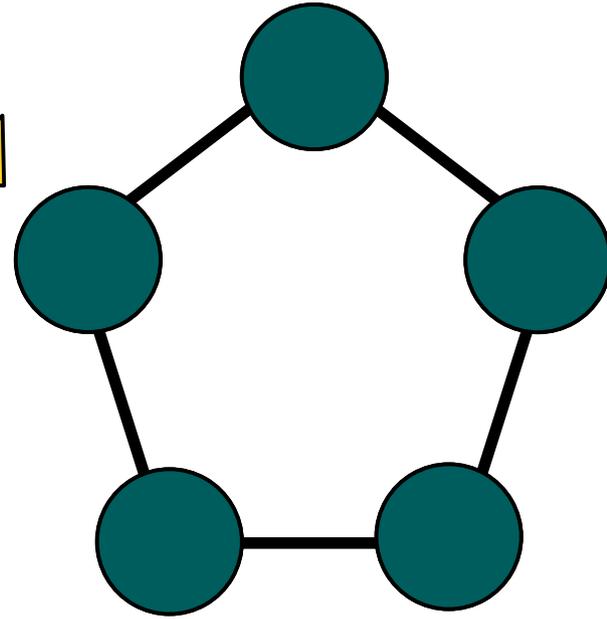
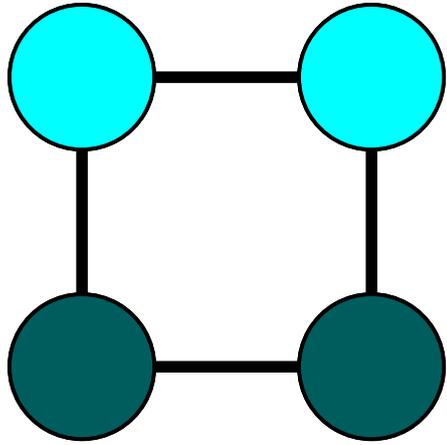
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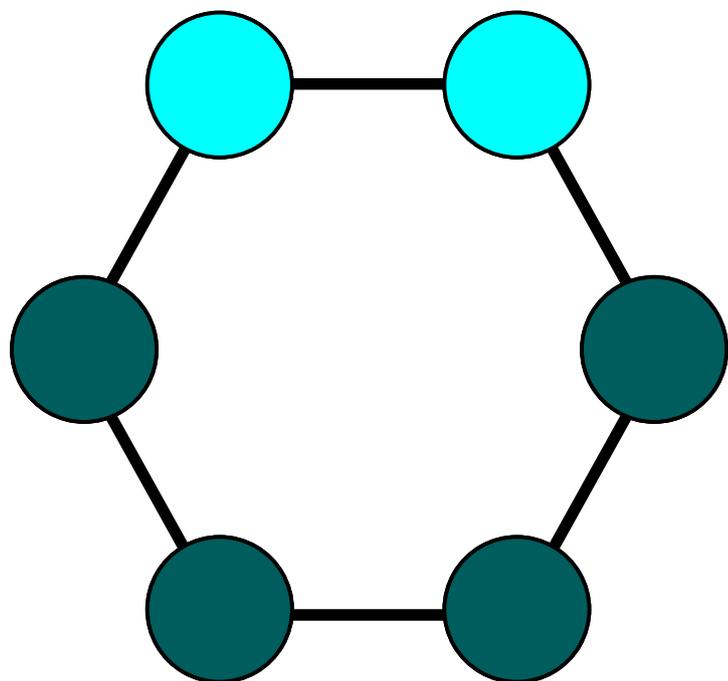
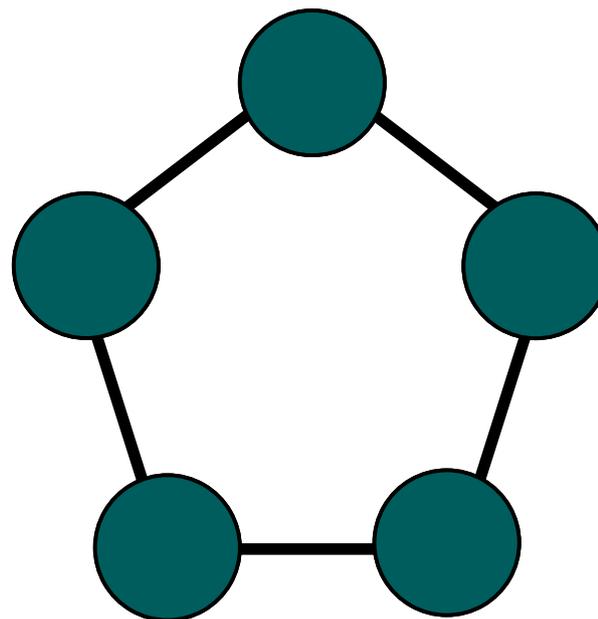
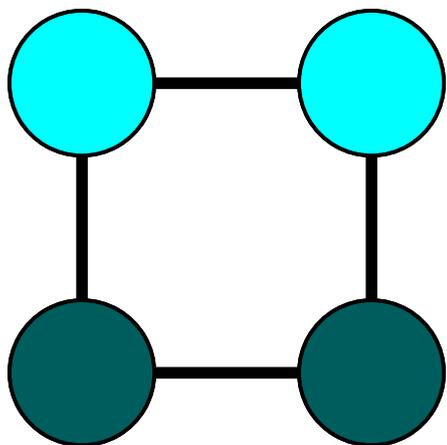
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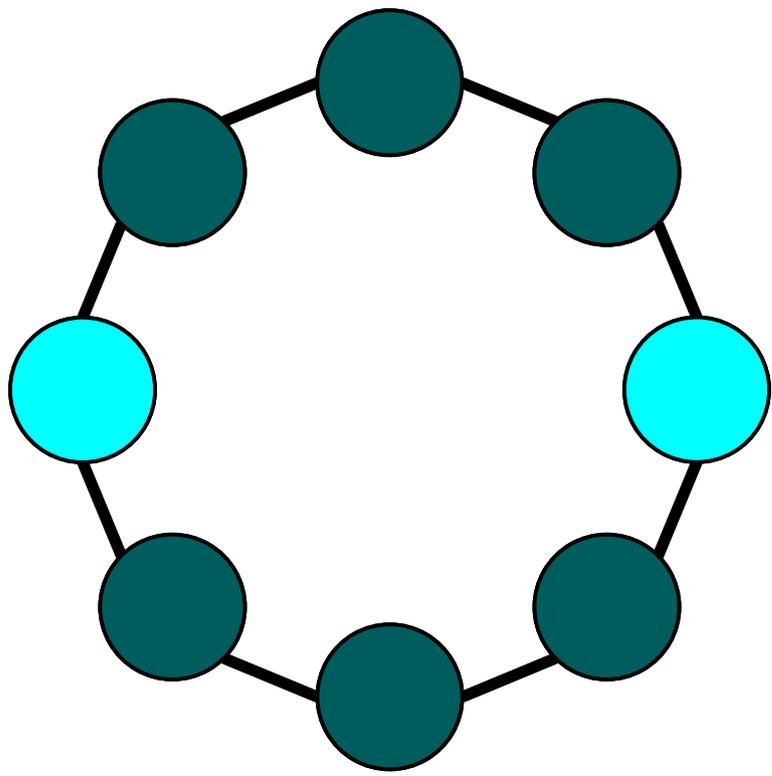


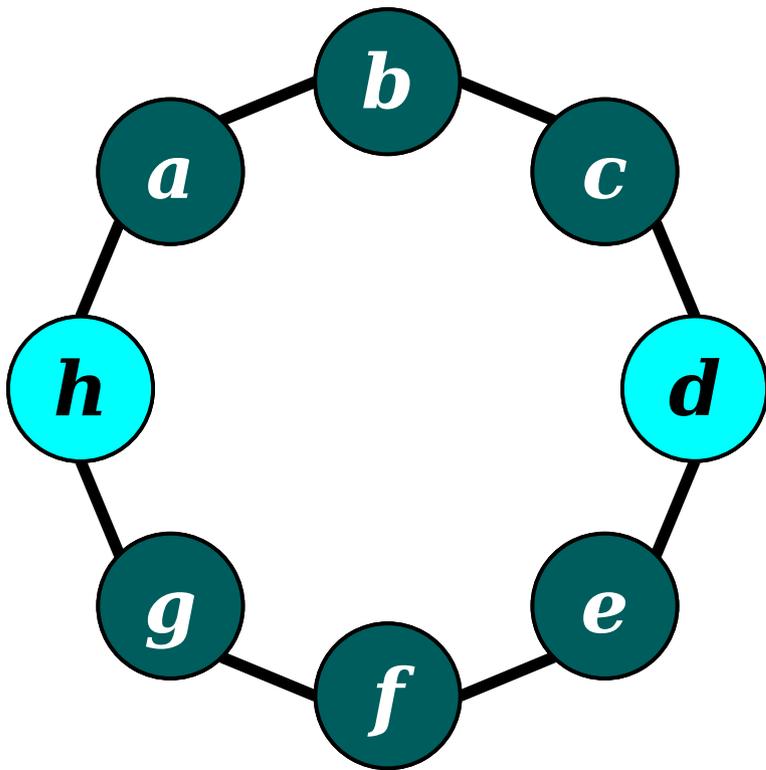






Solving Lights-Out With a SAT Solver





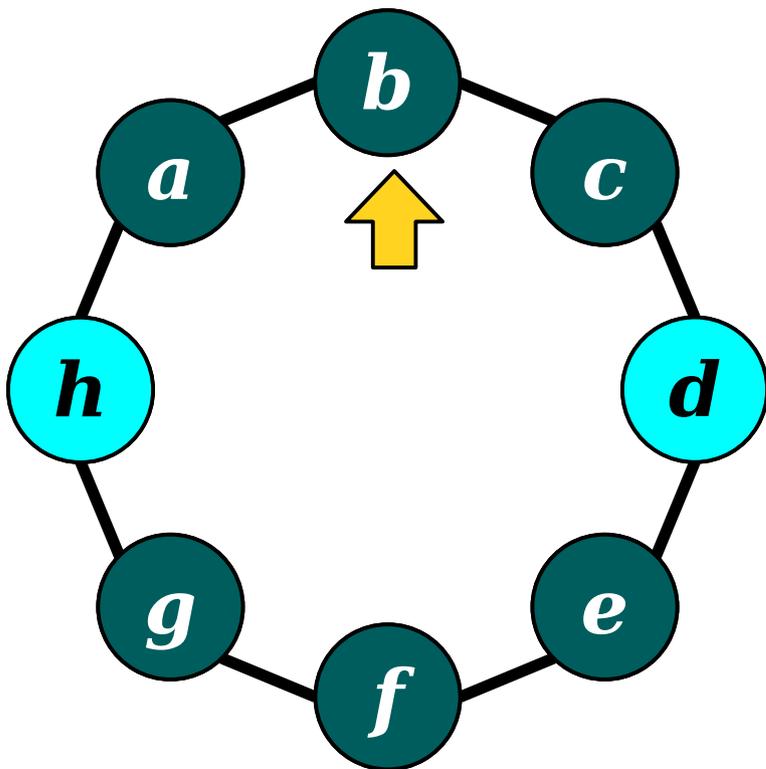
Observation 1: We never need to press the same button twice.

Observation 2: Button press order doesn't matter.

Observation 3: Make a propositional formula with one variable per button, indicating whether we press it.

Observation 4: A light that is initially off stays off when an even number of adjacent lights are pressed.

Observation 5: A light that is initially on ends off when an odd number of adjacent lights are pressed.



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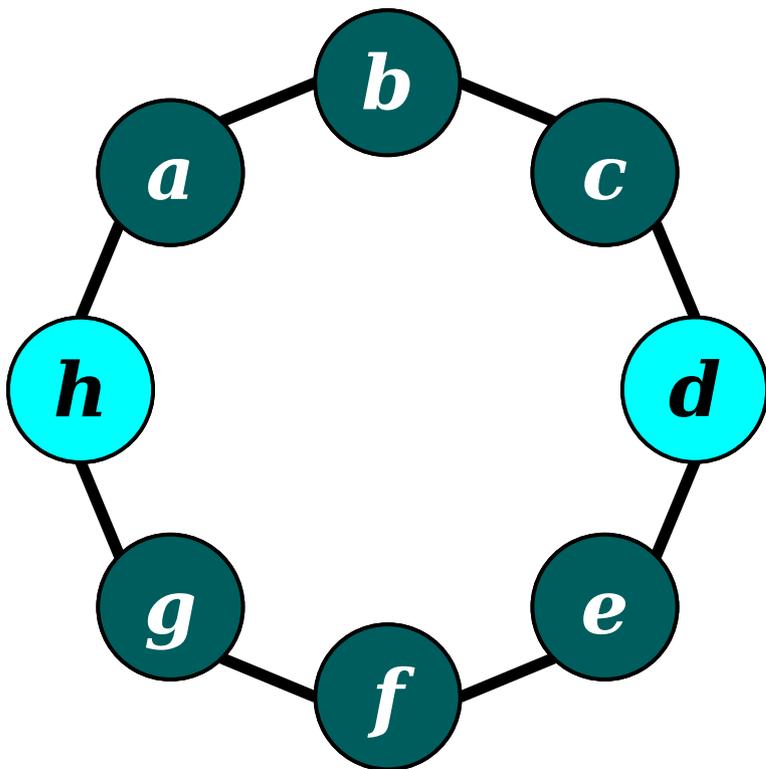
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Observation 5: A light that is initially on ends off when an odd number of adjacent lights are pressed.

Write a formula in propositional logic that says "an even number of the variables *a* and *c* are true."

Answer at

<https://cs103.stanford.edu/pollev>



$$\begin{aligned}
 &(h \leftrightarrow b) \wedge \\
 &(a \leftrightarrow c) \wedge \\
 &(b \leftrightarrow d) \wedge \\
 &\neg(c \leftrightarrow e) \wedge \\
 &(d \leftrightarrow f) \wedge \\
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 \end{aligned}$$

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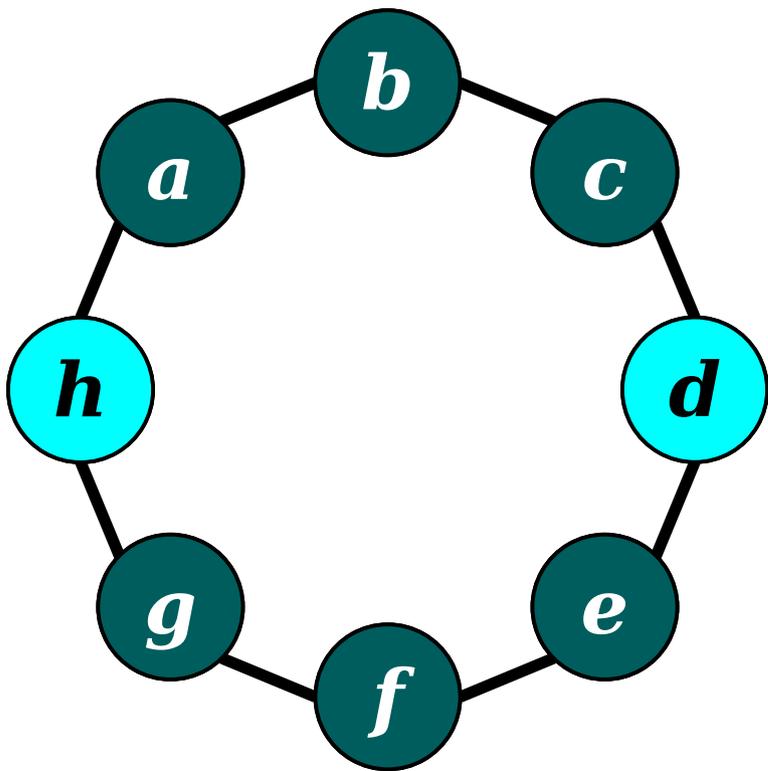
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Make a and c true.
Make $b, d, e, f, g,$ and h false.

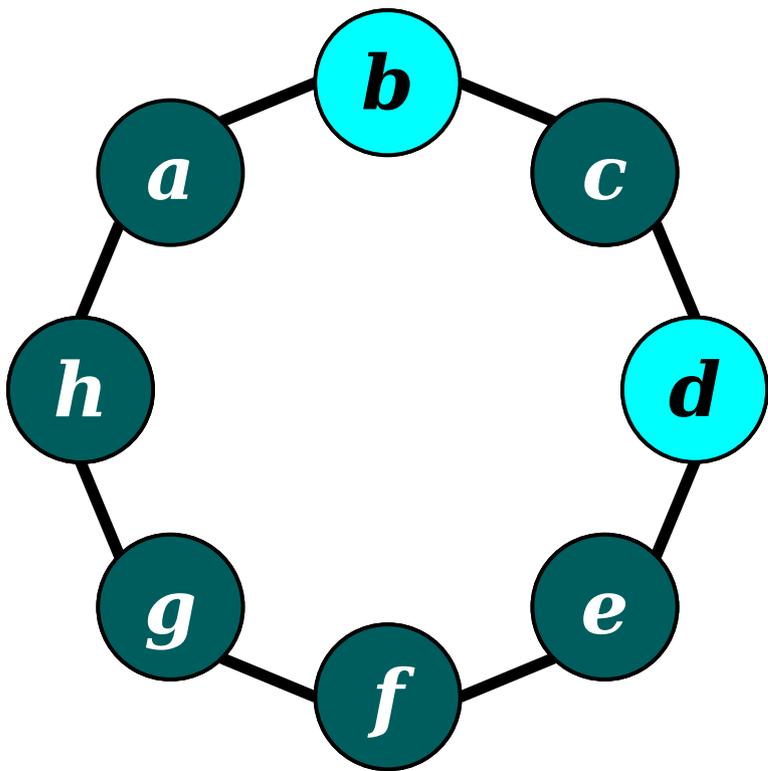
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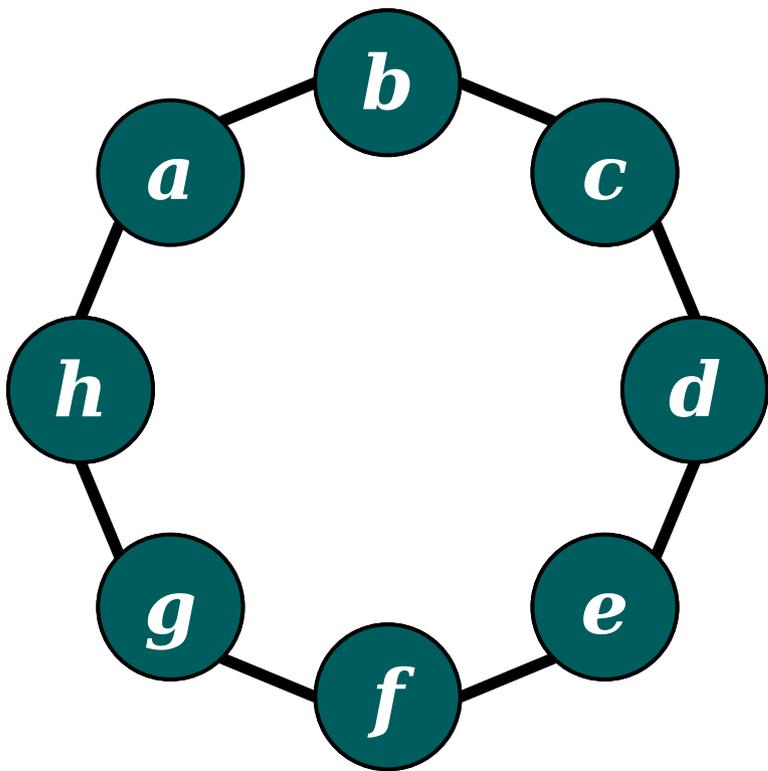
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Make a and c true.
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In Pseudocode

```
bool canTurnLightsOff(LightRing r) {  
    return isSatisfiable(ringToFormula(r));  
}
```

Intuition:

Solving Lights Out can't be "harder" than solving SAT because if we can solve SAT efficiently, we can solve Lights Out efficiently.

```
bool canPlaceDominoes(Grid G, int k) {  
    return hasMatching(gridToGraph(G), k);  
}
```

```
bool canTurnLightsOff(LightRing r) {  
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}
```

```
bool solveProblemA(string input) {  
    return solveProblemB(translate(input));  
}
```

Intuition:

Problem *A* can't be “harder” than problem *B*, because solving problem *B* lets us solve problem *A*.

```
bool solveProblemA(string input) {  
    return solveProblemB(translate(input));  
}
```

- If A and B are problems where it's possible to solve problem A using the strategy shown above*, we write

$$A \leq_p B.$$

- We say that ***A is polynomial-time reducible to B.***

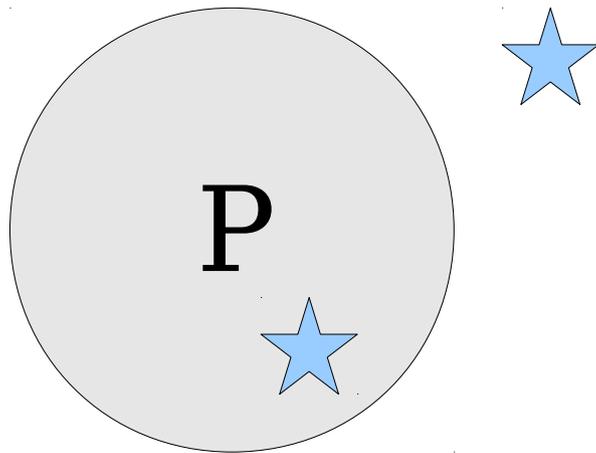
* Assuming that translate runs in polynomial time.

```
bool solveProblemA(string input) {  
    return solveProblemB(translate(input));  
}
```

- This is a powerful general problem-solving technique. You'll see it a lot in CS161.

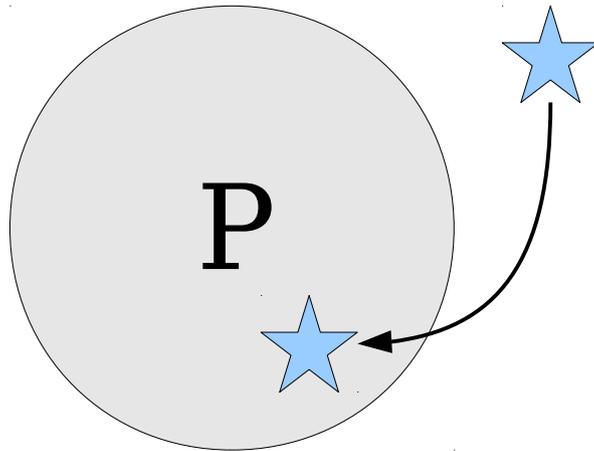
Polynomial-Time Reductions

- If $A \leq_p B$ and $B \in \mathbf{P}$, then $A \in \mathbf{P}$.



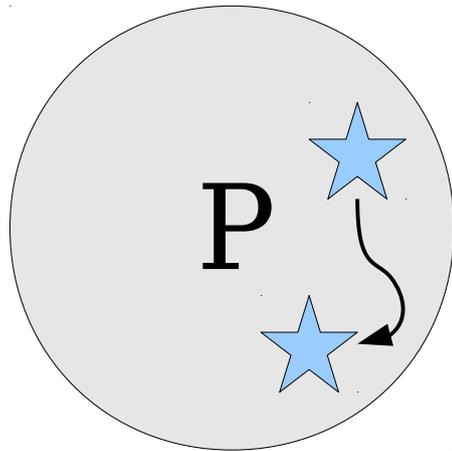
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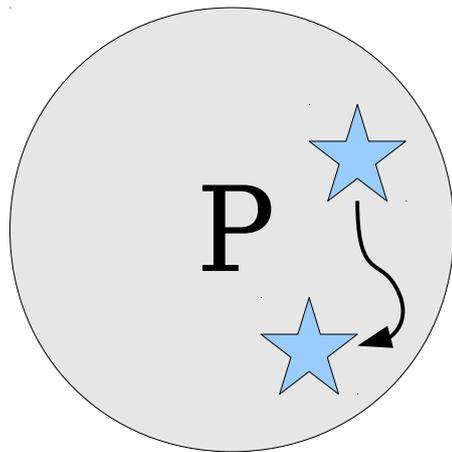
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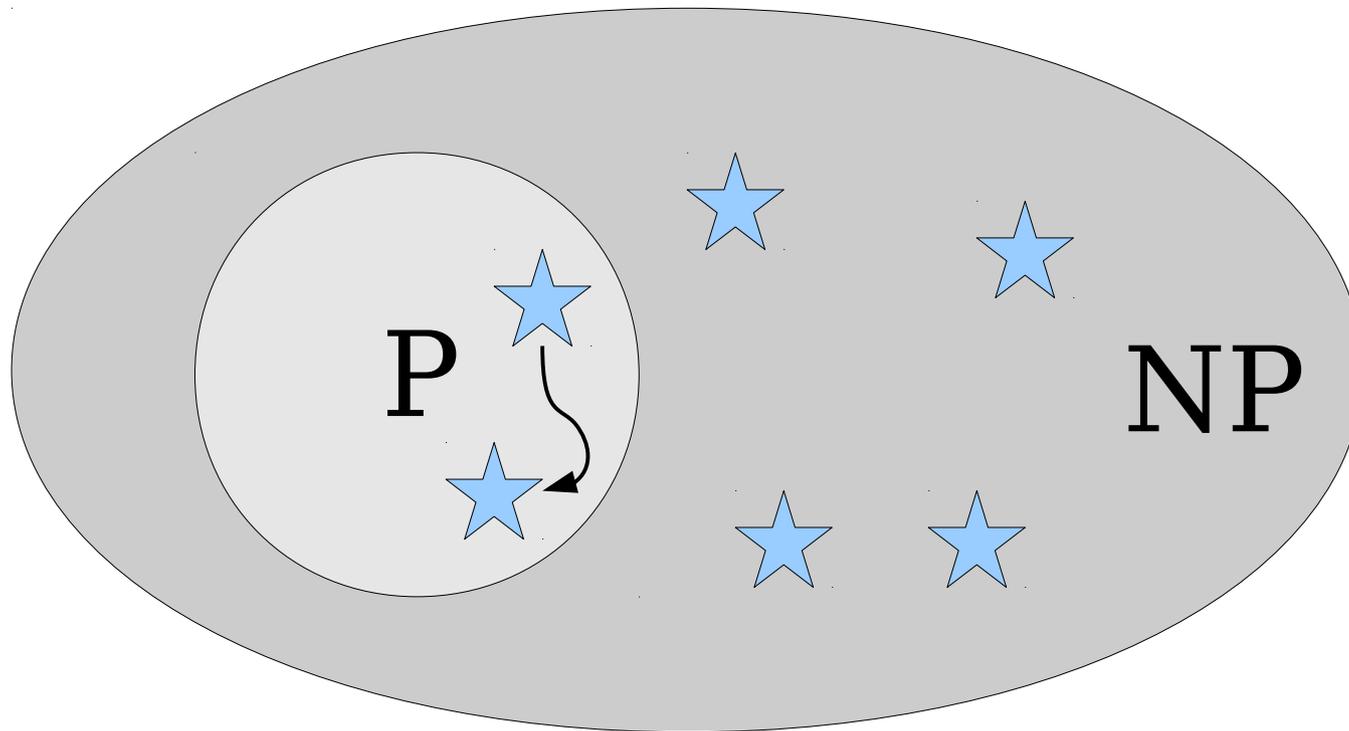
Polynomial-Time Reductions

- If $A \leq_p B$ and $B \in \mathbf{P}$, then $A \in \mathbf{P}$.
- If $A \leq_p B$ and $B \in \mathbf{NP}$, then $A \in \mathbf{NP}$.



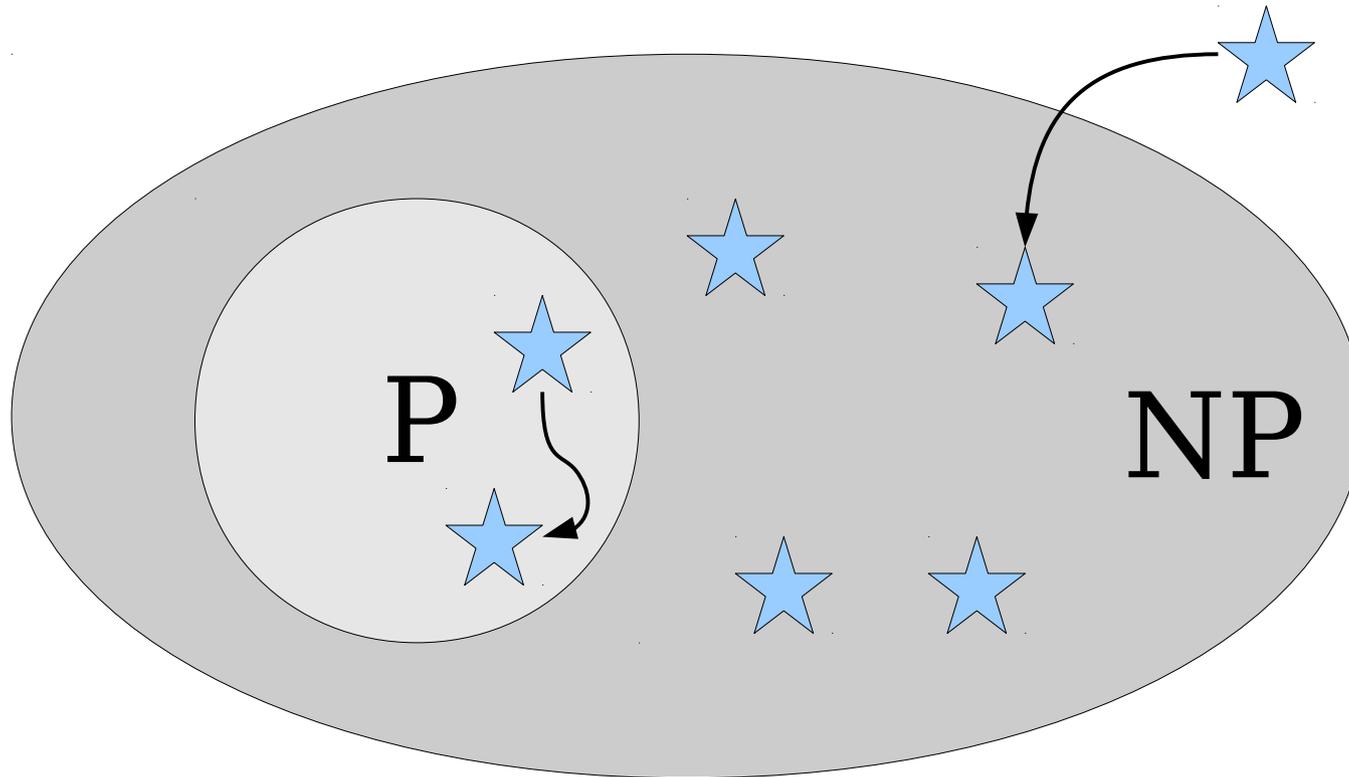
Polynomial-Time Reductions

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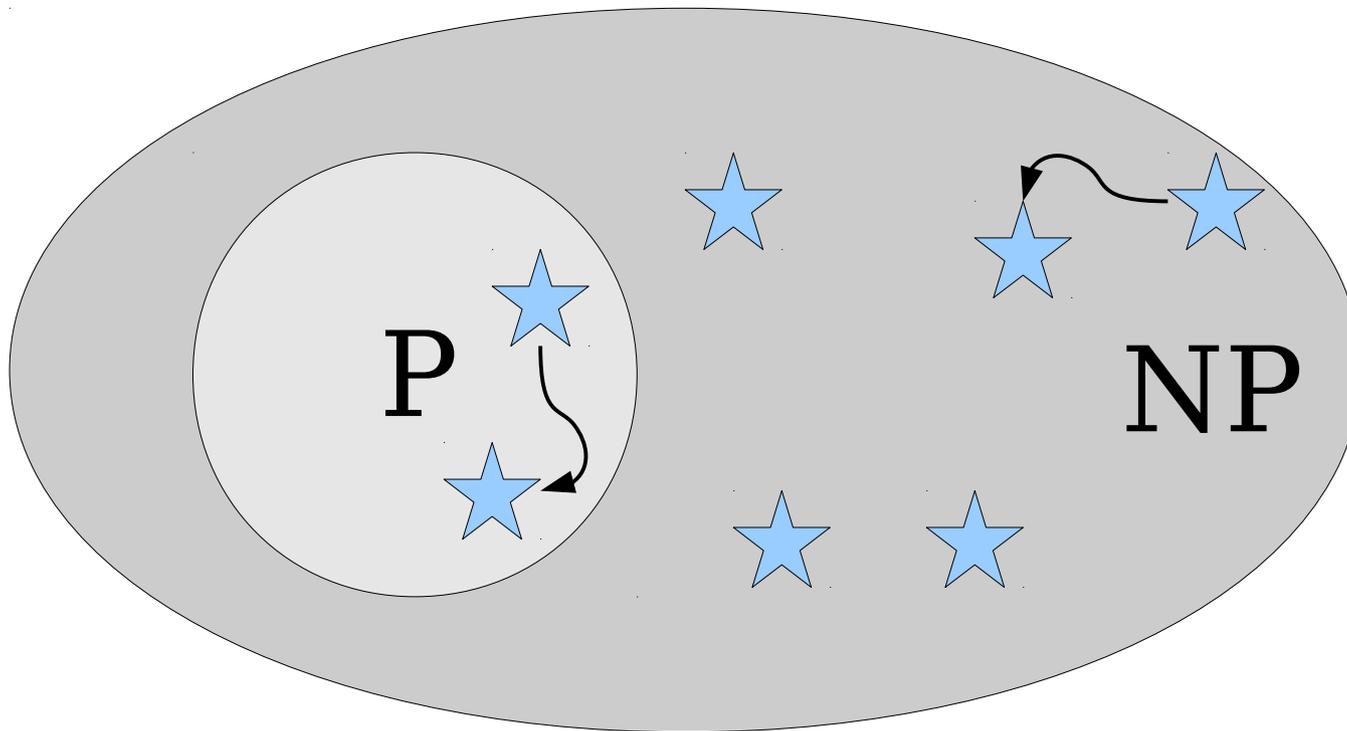
Polynomial-Time Reductions

- If $A \leq_p B$ and $B \in \mathbf{P}$, then $A \in \mathbf{P}$.
- If $A \leq_p B$ and $B \in \mathbf{NP}$, then $A \in \mathbf{NP}$.



Polynomial-Time Reductions

- If $A \leq_p B$ and $B \in \mathbf{P}$, then $A \in \mathbf{P}$.
- If $A \leq_p B$ and $B \in \mathbf{NP}$, then $A \in \mathbf{NP}$.



This \leq_p relation lets us rank the relative difficulties of problems in **P** and **NP**.

What else can we do with it?

Time-Out for Announcements!

Please evaluate this course on Axess.

Your feedback makes a difference.

Final Exam Logistics

- Our final exam is on ***Saturday, June 8th*** from ***8:30AM - 11:30AM***. It'll be held in ***Hewlett 200***.
- The final exam covers topics from PS0 – PS9 and L00 – L26. The format is similar to that of the midterm, with a mix of short-answer questions and formal written proofs.
- Like the midterms, it's closed-book, closed-computer, and limited-note. You can bring one double-sided 8.5" × 11" notes sheet with you.
- Students with OAE accommodations: at this point you should know the date/time/location of your alternate exam. If that isn't the case, contact us ASAP.

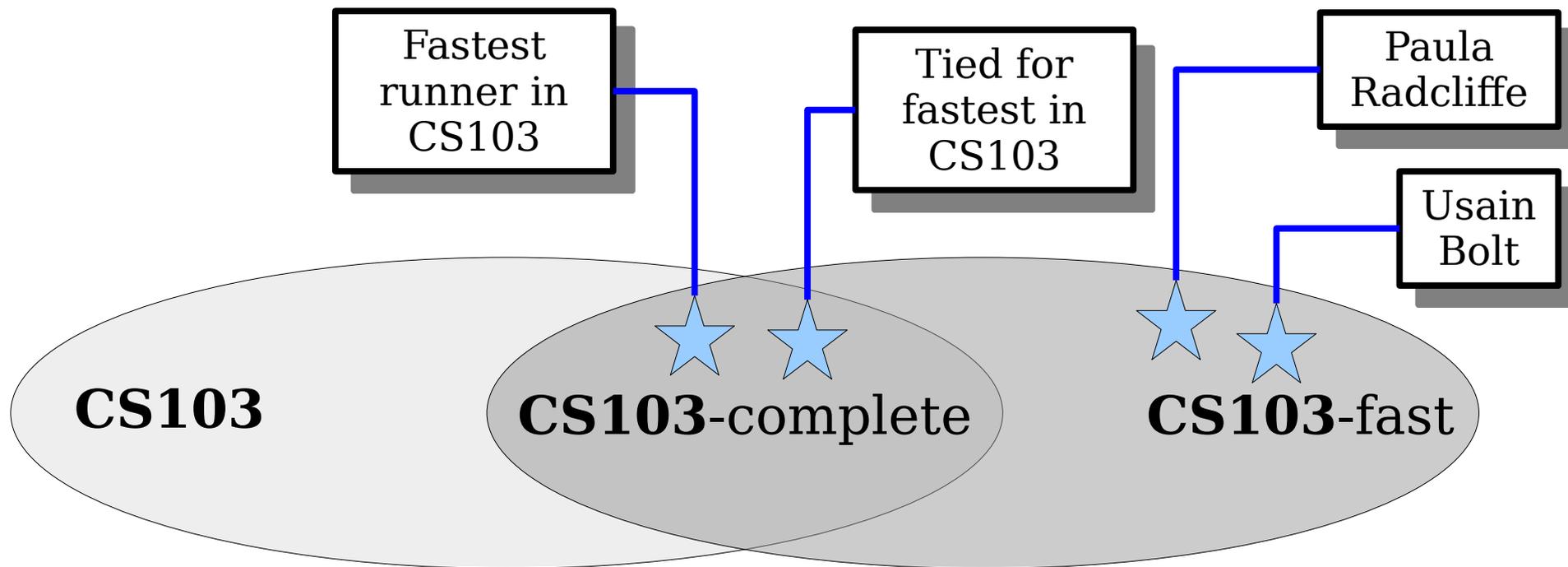
Preparing for the Exam

- Stanley will be holding a review session this **Thursday, June 6th** from **7:30PM - 8:30PM** in Hewlett 103.
- We've also released EPP3, a collection of four practice final exams you can use to prepare.
- We've also released the Cumulative Practice Problems list, a gigantic searchable database of problems you can use to brush up on whatever topics you need the most practice with.
- As always, **keep the TAs in the loop when studying!** That's what we're here for.

Back to CS103!

NP-Hardness and **NP**-Completeness

An Analogy: Running Really Fast



For people A and B , we say $A \leq_r B$ if A 's top running speed is at most B 's top speed.
(Intuitively: B can run at least as fast as A .)

We say that person P is **CS103-fast** if

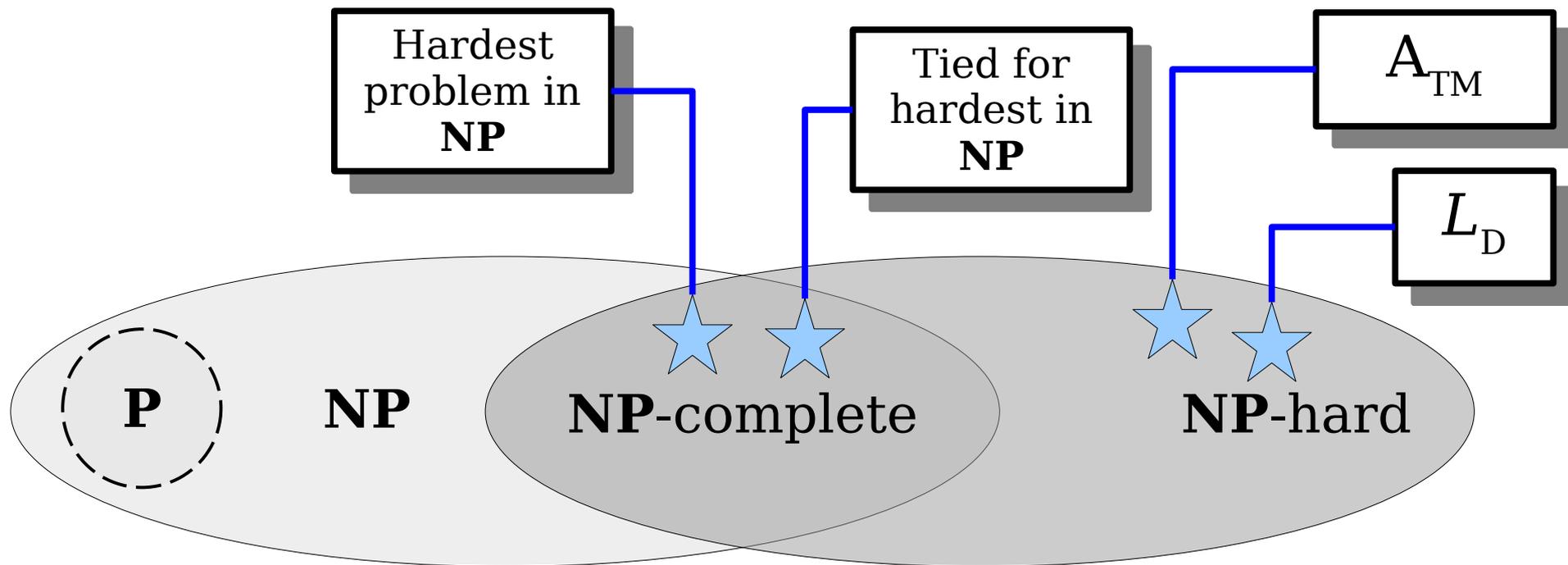
$$\forall A \in \mathbf{CS103}. A \leq_r P.$$

(How fast are you if you're CS103-fast?)

We say that person P is **CS103-complete** if

$$P \in \mathbf{CS103} \text{ and } P \text{ is } \mathbf{CS103-fast}.$$

(How fast are you if you're CS103-complete?)



For languages A and B , we say $A \leq_p B$ if A reduces to B in polynomial time.

(Intuitively: B is at least as hard as A .)

We say that a language L is ***NP-hard*** if

$$\forall A \in \mathbf{NP}. A \leq_p L.$$

(How hard is a problem that's NP-hard?)

We say that a language L is ***NP-complete*** if

$$L \in \mathbf{NP} \text{ and } L \text{ is } \mathbf{NP}\text{-hard}.$$

(How hard is a problem that's NP-complete?)

Intuition: The **NP**-complete problems are the hardest problems in **NP**.

If we can determine how hard those problems are, it would tell us a lot about the **P** $\stackrel{?}{=}$ **NP** question.

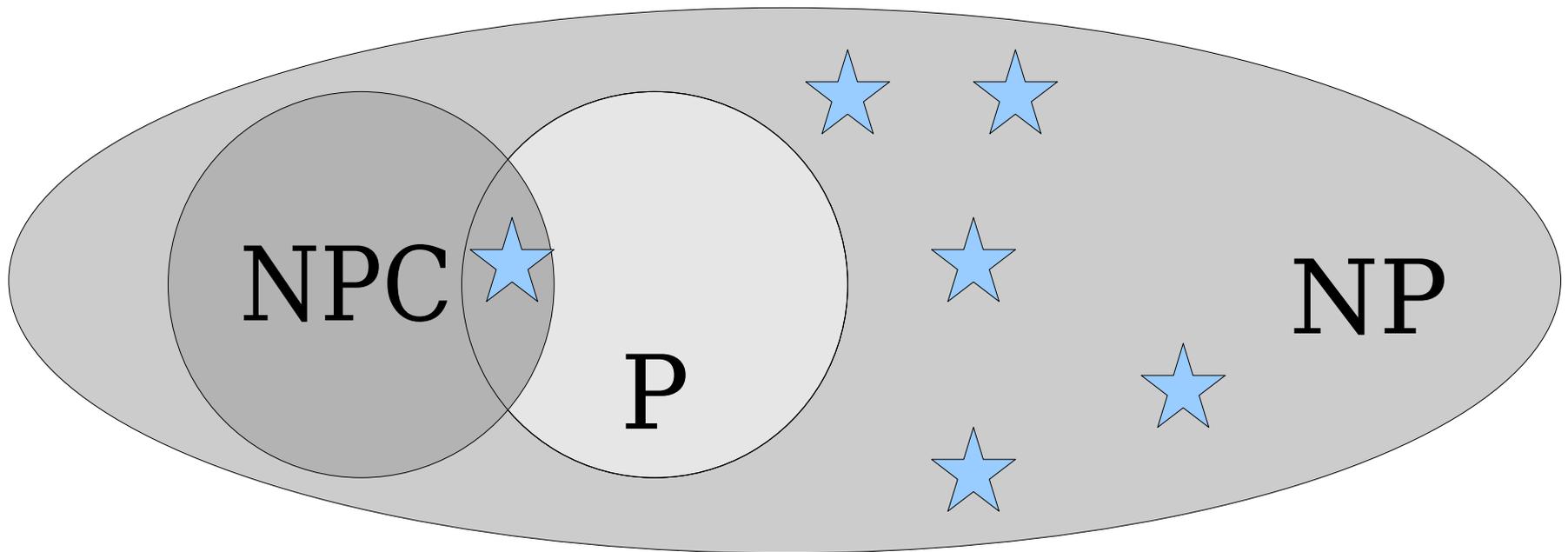
The Tantalizing Truth

Theorem: If *any* **NP**-complete language is in **P**, then **P** = **NP**.

Intuition: This means the hardest problems in **NP** aren't actually that hard. We can solve them in polynomial time. So that means we can solve all problems in **NP** in polynomial time.

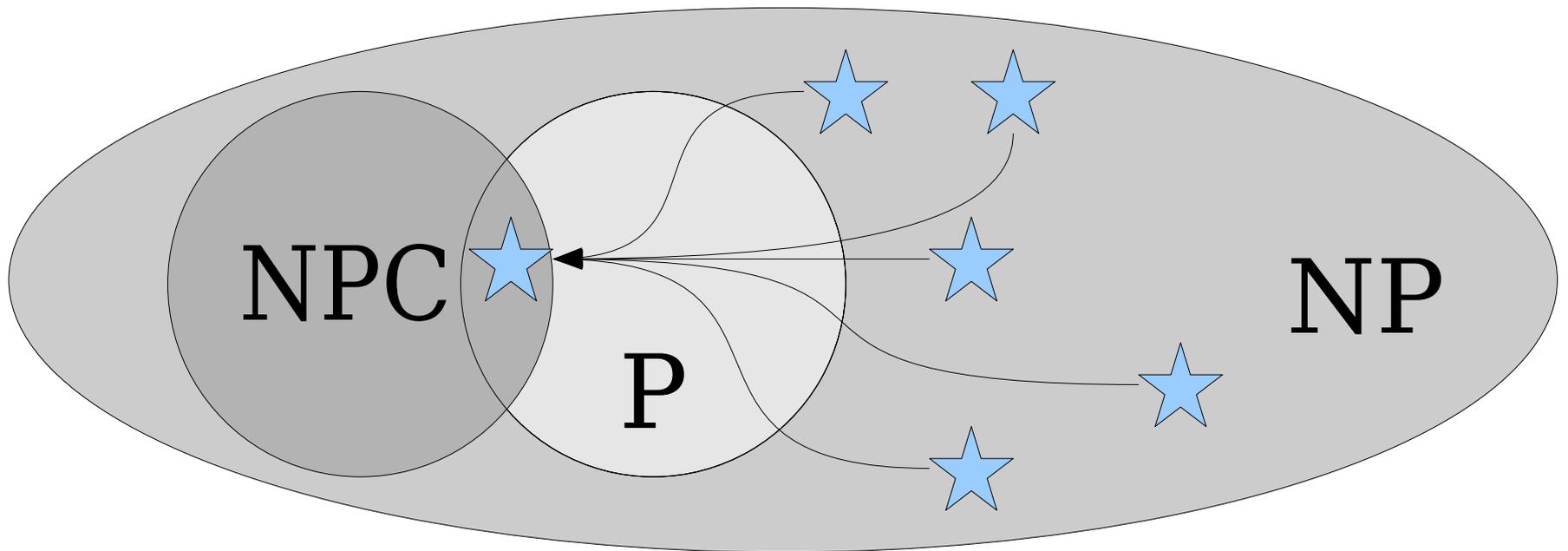
The Tantalizing Truth

Theorem: If *any* NP-complete language is in \mathbf{P} , then $\mathbf{P} = \mathbf{NP}$.



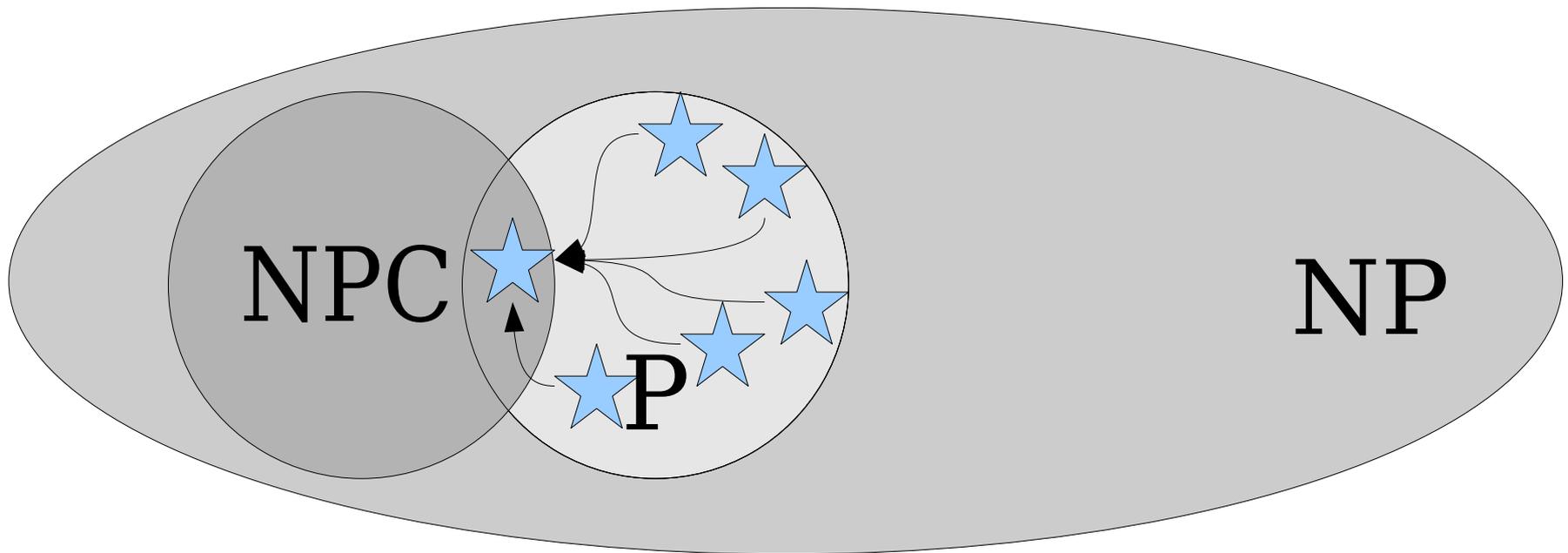
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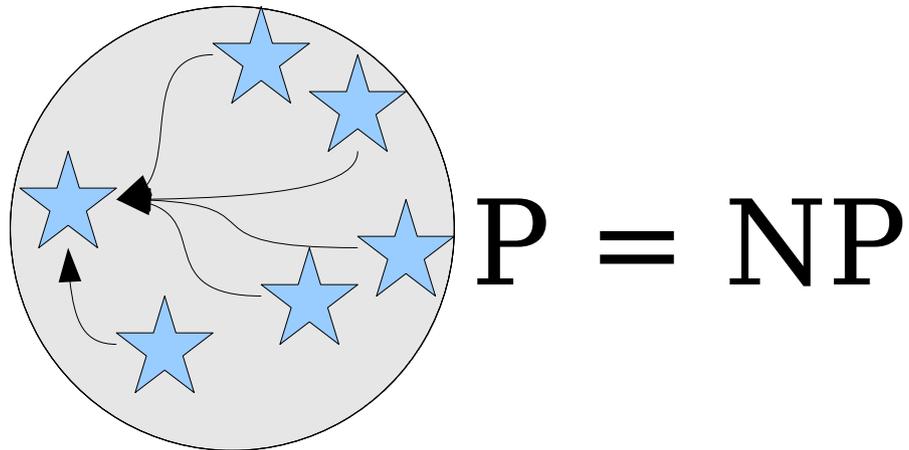
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The Tantalizing Truth

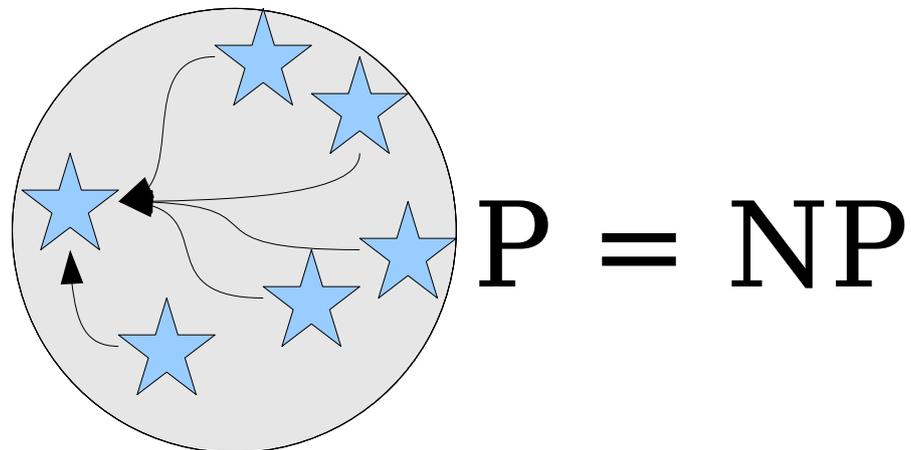
Theorem: If *any* NP-complete language is in **P**, then **P** = **NP**.



The Tantalizing Truth

Theorem: If *any* **NP**-complete language is in **P**, then **P** = **NP**.

Proof: Suppose that L is **NP**-complete and $L \in \mathbf{P}$. Now consider any arbitrary **NP** problem A . Since L is **NP**-complete, we know that $A \leq_p L$. Since $L \in \mathbf{P}$ and $A \leq_p L$, we see that $A \in \mathbf{P}$. Since our choice of A was arbitrary, this means that $\mathbf{NP} \subseteq \mathbf{P}$, so **P** = **NP**. ■



The Tantalizing Truth

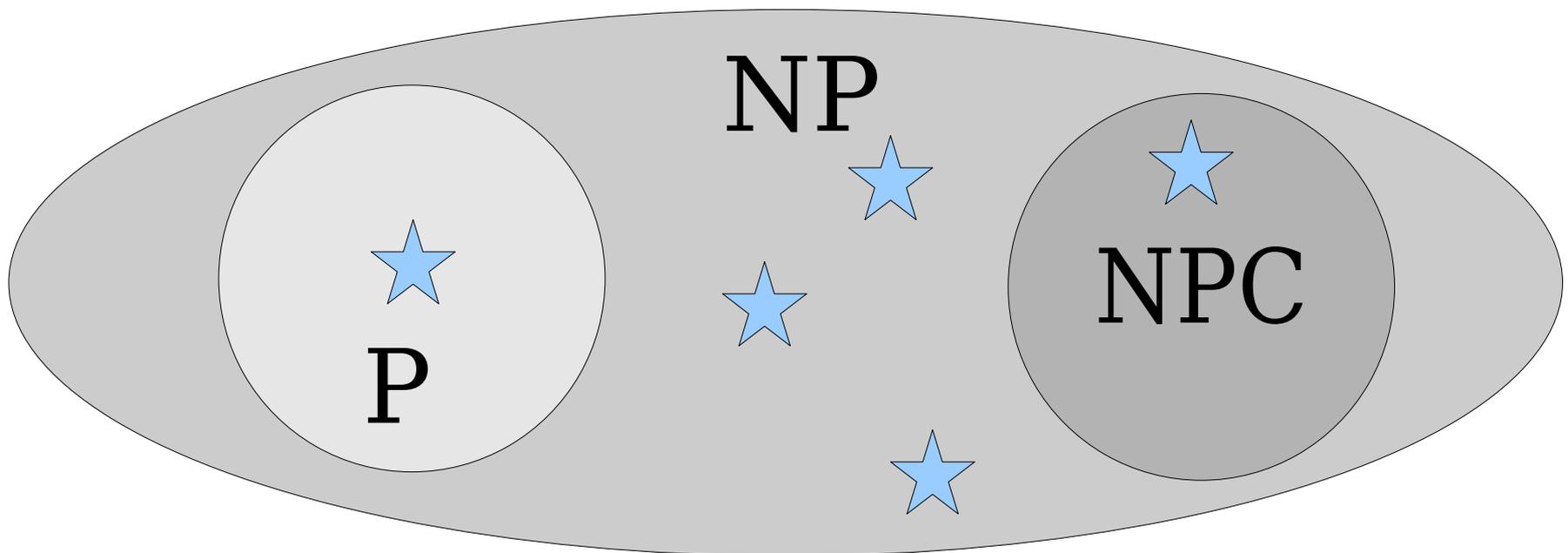
Theorem: If *any* **NP**-complete language is not in **P**, then **P** \neq **NP**.

Intuition: This means the hardest problems in **NP** are so hard that they can't be solved in polynomial time. So the hardest problems in **NP** aren't in **P**, meaning **P** \neq **NP**.

The Tantalizing Truth

Theorem: If *any* **NP**-complete language is not in **P**, then **P** \neq **NP**.

Proof: Suppose that L is an **NP**-complete language not in **P**. Since L is **NP**-complete, we know that $L \in \mathbf{NP}$. Therefore, we know that $L \in \mathbf{NP}$ and $L \notin \mathbf{P}$, so **P** \neq **NP**. ■



How do we even know NP-complete problems exist in the first place?

Theorem (Cook-Levin): SAT is **NP**-complete.

Proof Idea: To see that **SAT** \in **NP**, show how to make a polynomial-time verifier for it. Key idea: have the certificate be a satisfying assignment.

To show that **SAT** is **NP**-hard, given a polynomial-time verifier V for an arbitrary **NP** language L , for any string w you can construct a polynomially-sized formula $\varphi(w)$ that says “there is a certificate c where V accepts $\langle w, c \rangle$.” This formula is satisfiable if and only if $w \in L$, so deciding whether the formula is satisfiable decides whether w is in L . ■

Proof: Take CS154!

Why All This Matters

- Resolving $\mathbf{P} \stackrel{?}{=} \mathbf{NP}$ is equivalent to just figuring out how hard SAT is.

$$\mathbf{SAT} \in \mathbf{P} \quad \leftrightarrow \quad \mathbf{P} = \mathbf{NP}$$

- We've turned a huge, abstract, theoretical problem about solving problems versus checking solutions into the concrete task of seeing how hard one problem is.
- You can get a sense for how little we know about algorithms and computation given that we can't yet answer this question!

Why All This Matters

- You will almost certainly encounter **NP**-hard problems in practice – they're everywhere!
- If a problem is **NP**-hard, then there is no known algorithm for that problem that
 - is efficient on all inputs,
 - always gives back the right answer, and
 - runs deterministically.
- ***Useful intuition:*** If you need to solve an **NP**-hard problem, you will either need to settle for an approximate answer, an answer that's likely but not necessarily right, or have to work on really small inputs.

Sample NP-Hard Problems

- **Computational biology:** Given a set of genomes, what is the most probable evolutionary tree that would give rise to those genomes? (*Maximum parsimony problem*)
- **Game theory:** Given an arbitrary perfect-information, finite, two-player game, who wins? (*Generalized geography problem*)
- **Operations research:** Given a set of jobs and workers who can perform those tasks in parallel, can you complete all the jobs within some time bound? (*Job scheduling problem*)
- **Machine learning:** Given a set of data, find the simplest way of modeling the statistical patterns in that data. (*Bayesian network inference problem*)
- **Medicine:** Given a group of people who need kidneys and a group of kidney donors, find the maximum number of people who can receive transplants. (*Cycle cover problem*)
- **Systems:** Given a set of processes and a number of processors, find the optimal way to assign those tasks so that they complete as soon as possible. (*Processor scheduling problem*)

Coda: What if $\mathbf{P} \stackrel{?}{=} \mathbf{NP}$ is resolved?

Next Time

- ***Why All This Matters***
- ***Where to Go from Here***
- ***A Final “Your Questions”***
- ***Parting Words!***